FINAL JEE(Advanced) EXAMINATION - 2020

(Held On Sunday 27th SEPTEMBER, 2020)

PAPER-1 | **TEST PAPER WITH SOLUTION**

PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks : 18)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: +3	If ONLY the correct option is chosen;
Zero Marks	: 0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	: –1	In all other cases.

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1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c,d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of

$$ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$$

is

(A) 0 (B) 8000 (C) 8080 (D) 16000

Ans. **(D)**

Sol. $x^2 + 20x - 2020 = 0$ has two roots $a,b \in R$

$x^2 - 20x + 2020 = 0$ has two roots $c,d \in$ complex

$ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$

$= a^2c - ac^2 + a^2d - ad^2 + b^2c - bc^2 + b^2d - bd^2$

$= a^2(c+d) + b^2(c+d) - c^2(a+b) - d^2(a+b)$

$= (c+d)(a^2+b^2) - (a+b)(c^2+d^2)$

$= (c+d)((a+b)^2 - 2ab) - (a+b)((c+d)^2 - 2cd)$

$= 20[(20)^2 + 4040] + 20[(20)^2 - 4040]$

$= 20[(20)^2 + 4040 + (20)^2 - 4040]$

$= 20 \times 800 = 16000$

2. If the function $f:\mathbb{R}\to\mathbb{R}$ is defined by f(x) = |x|(x – sinx), then which of the following statements is **TRUE ?**

(A) f is one-one, but **NOT** onto

(B) f is onto, but **NOT** one-one

(C) f is **BOTH** one-one and onto

(D) f is **NEITHER** one-one **NOR** onto

Ans. (C)

Sol. f(x) is a non-periodic, continuous and odd function

$$f(x)=\begin{cases}-x^2+x\sin x, & x<0\\ x^2-x\sin x, & x\ge 0\end{cases}$$

$$f(-\infty)=\operatorname*{Lt}_{x\to-\infty}(-x^2)\left(1-\frac{\sin x}{x}\right)=-\infty$$

$$f(\infty)=\operatorname*{Lt}_{x\to\infty}x^2\left(1-\frac{\sin x}{x}\right)=\infty$$

$\Rightarrow$ Range of f(x) = R

$\Rightarrow$ f(x) is an onto function…(1)

$$f'(x)=\begin{cases}-2x+\sin x+x\cos x, & x<0\\ 2x-\sin x-x\cos x, & x\ge 0\end{cases}$$

For (0, ∞)

$$f'(x)=\underbrace{(x-\sin x)}_{\substack{\text{always +ve}\\ \text{or } 0}}+\underbrace{x(1-\cos x)}_{\substack{\text{always +ve}\\ \text{or } 0}}$$

$\Rightarrow$ $f'(x)>0$

$\Rightarrow$ $f'(x)\ge 0,\ \forall x\in(-\infty,\infty)$

equality at x = 0

$\Rightarrow$ f(x) is one –one function …. (2)

From (1) & (2), f(x) is both one-one & onto.

3. Let the functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be defined by

$f(x)=e^{x-1}-e^{-|x-1|}$ and $g(x)=\frac{1}{2}\left(e^{x-1}+e^{1-x}\right)$.

Then the area of the region in the first quadrant bounded by the curves y = f(x), y = g(x) and x = 0 is

(A) $\left(2-\sqrt{3}\right)+\frac{1}{2}(e-e^{-1})$

(B) $\left(2+\sqrt{3}\right)+\frac{1}{2}(e-e^{-1})$

(C) $\left(2-\sqrt{3}\right)+\frac{1}{2}(e+e^{-1})$

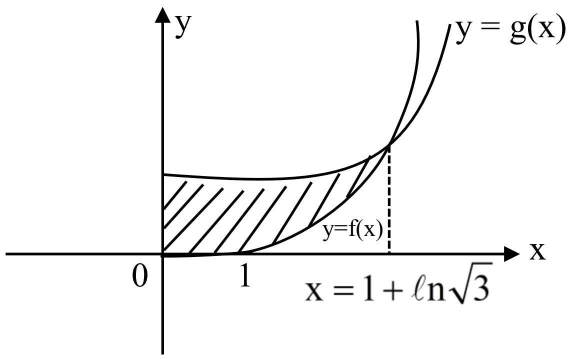
(D) $\left(2+\sqrt{3}\right)+\frac{1}{2}(e+e^{-1})$

Ans. (A)

Sol. Here

$$f(x)=\begin{cases} 0 & x\le 1 \\ e^{x-1}-e^{1-x} & x\ge 1 \end{cases}$$

$$\& \;\; g(x)=\frac{1}{2}\left(e^{x-1}+e^{1-x}\right)$$



solve f(x) & g(x) $\Rightarrow x = 1+\ell n\sqrt{3}$

So bounded area = $\int_0^1 \frac{1}{2}\left(e^{x-1}+e^{1-x}\right)dx+\int_1^{1+\ell n\sqrt{3}} \frac{1}{2}\left(e^{x-1}+e^{1-x}\right)-\left(e^{x-1}+e^{1-x}\right)dx$

$$=\frac{1}{2}\left[e^{x-1}-e^{1-x}\right]_0^1+\left[-\frac{1}{2}e^{x-1}-\frac{3}{2}e^{1-x}\right]_1^{1+\ell n\sqrt{3}}$$

$$=\frac{1}{2}\left[e-\frac{1}{e}\right]+\left[\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)+2\right]=2-\sqrt{3}+\frac{1}{2}\left(e-\frac{1}{e}\right)$$

Ans. (A)

4. Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$

Ans. (A)

Sol. $y^2 = 4\lambda x$, $P(\lambda, 2\lambda)$

Slope of the tangent to the parabola at point P

$$\frac{dy}{dx}=\frac{4\lambda}{2y}=\frac{4\lambda}{2x2\lambda}=1$$

Slope of the tangent to the ellipse at P

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$$

As tangents are perpendicular $y' = -1$

$$\Rightarrow \frac{2\lambda}{a^2} - \frac{4\lambda}{b^2} = 0 \Rightarrow \frac{a^2}{b^2} = \frac{1}{2}$$

$$\Rightarrow e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Ans. A

5. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently, Then probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

(A) $\frac{40}{81}$ (B) $\frac{20}{81}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Ans. (B)

Sol. $P(H) = \frac{2}{3}$ for C_1

$P(H) = \frac{1}{3}$ for C_2

for C_1

No. of Heads (α)	0	1	2
Probability	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$

for C_2

No. of Heads (β)	0	1	2
Probability	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

for real and equal roots

$\alpha^2 = 4\beta$

$(\alpha, \beta) = (0, 0), (2, 1)$

So, probability $= \frac{1}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{4}{9} = \frac{20}{81}$

6. Consider all rectangles lying in the region

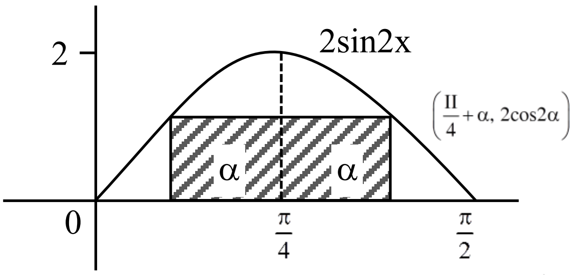
$$\left\{(x,y)\in\mathbb{R}\times\mathbb{R} : 0\le x\le \frac{\pi}{2} \text{ and } 0\le y\le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

(A) $\frac{3\pi}{2}$ (B) π (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

Ans. (C)

Sol.



Perimeter = $2(2\alpha + 2\cos 2\alpha)$

$P = 4(\alpha + \cos 2\alpha)$

$\frac{dP}{d\alpha} = 4(1 - 2\sin 2\alpha) = 0$

$\sin 2\alpha = \frac{1}{2}$

$2\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$

$\frac{d^2P}{d\alpha^2} = -4\cos 2\alpha$

for maximum $\alpha = \frac{\pi}{12}$

Area = $(2\alpha)(2\cos 2\alpha)$

$= \frac{\pi}{6} \times 2 \times \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$

SECTION-2 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: +4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	: +3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	: 0	If none of the options is chose (i.e. the question is unanswered);
Negative Marks	: –2	In all other cases

7. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x-1)\sin x$ and let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Let $fg : \mathbb{R} \to \mathbb{R}$ be the product function defined by $(f g)(x) = f(x)\, g(x)$. Then which of the following statements is/are TRUE ?

(A) If g is continuous at x = 1, then fg is differentiable at x = 1

(B) If fg is differentiable at x = 1, then g is continuous at x = 1

(C) If g is differentiable at x = 1, then fg is differentiable at x = 1

(D) If fg is differentiable at x = 1, then g is differentiable at x = 1

Ans. (A,C)

Sol. $f : R \to R \quad f(x) = (x^2 + \sin x)(x-1) \quad f(1^+) = f(1^-) = f(1) = 0$

$fg(x) : f(x).g(x) \quad fg : R \to R$

let $fg(x) = h(x) = f(x).g(x) \quad h:R\to R$

option (c) $h'(x) = f'(x)g(x) + f(x)\,g'(x)$

$h'(1) = f'(1)\,g(1) + 0,$

(as $f(1) = 0$, $g'(x)$ exists}

$\Rightarrow$ if g(x) is differentiable then h(x) is also differentiable (true)

option (A) If g(x) is continuous at x = 1 then $g(1^+) = g(1^-) = g(1)$

$$h'(1^+) = \lim_{h\to 0^+} \frac{h(1+h) - h(1)}{h}$$

$$h'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)g(1+h) - 0}{h} = f'(1)g(1)$$

$$h'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)g(1-h) - 0}{-h} = f'(1)g(1)$$

So $h(x) = f(x).g(x)$ is differentiable

at x = 1 (True)

option (B) (D) $h'(1^+) = \lim_{h\to 0^+} \frac{h(1+h)-h(1)}{-h}$

$$h'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)g(1+h)}{h} = f'(1)g(1^+)$$

$$h'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)g(1-h)}{-h} = f'(1).g(1^-)$$

$$\Rightarrow g(1^+) = g(1^-)$$

So we cannot comment on the continuity and differentiability of the function.

8. Let M be a 3 × 3 invertible matrix with real entries and let I denote the 3 × 3 identity matrix. If M^{-1} = adj (adj M), then which of the following statement is/are ALWAYS TRUE ?

(A) M = I (B) det M = 1 (C) $M^2 = I$ (D) $(\text{adj } M)^2 = I$

Ans. (B,C,D)

Sol. det (M) ≠ 0

$M^{-1} = \text{adj}(\text{adj } M)$

$M^{-1} = \det(M).M$

$M^{-1}M = \det(M).M^2$

$I = \det(M).M^2$ …. (i)

$\det(I) = (\det(M))^5$

$1 = \det(M)$ …. (ii)

From (i) $I = M^2$

$(\text{adj } M)^2 = \text{adj } (M^2) = \text{adj } I = I$

9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE ?

(A) $\left|z+\frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$

(B) $|z| \le 2$ for all $z \in S$

(C) $\left|z+\frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$

(D) The set S has exactly four elements

Ans. (B,C)

Sol. $|z^2 + z + 1| = 1$

$$\Rightarrow \left|\left(z+\frac{1}{2}\right)^2 + \frac{3}{4}\right| = 1$$

$$\Rightarrow \left|\left(z+\frac{1}{2}\right)^2 + \frac{3}{4}\right| \le \left|z+\frac{1}{2}\right|^2 + \frac{3}{4}$$

$$\Rightarrow 1 \le \left|z+\frac{1}{2}\right|^2 + \frac{3}{4} \quad \Rightarrow \left|\left(z+\frac{1}{2}\right)\right|^2 \ge \frac{1}{4}$$

$$\Rightarrow \left|z+\frac{1}{2}\right| \ge \frac{1}{2}$$

also $|(z^2 + z) + 1| = 1 \ge ||z^2 + z| - 1|$

$\Rightarrow |z^2 + z| - 1 \le 1$

$\Rightarrow |z^2 + z| \le 2$

$\Rightarrow ||z^2| - |z|| \le |z^2 + z| \le 2$

$\Rightarrow |r^2 - r| \le 2$

$\Rightarrow \quad r = |z| \le 2 \; ; \forall\, z \in S$

Also we can always find root of the equation $z^2 + z + 1 = e^{i\theta}$; $\forall\, \theta \in R$

Hence set 'S' is infinite

10. Let x, y and z be positive real numbers. Suppose x, y and z are lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE?

(A) $2Y = X + Z$

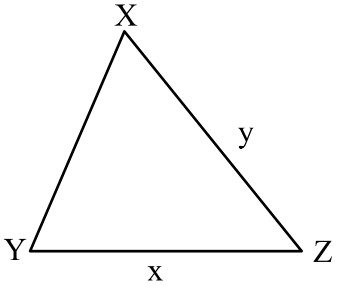
(B) $Y = X + Z$

(C) $\tan\frac{X}{2} = \frac{x}{y+z}$

(D) $x^2 + z^2 - y^2 = xz$

Ans. (B,C)

Sol.



$$\tan\frac{x}{2}+\tan\frac{z}{2}=\frac{2y}{x+y+z}$$

$$\frac{\Delta}{S(S-x)}+\frac{\Delta}{S(S-z)}=\frac{2y}{2S}$$

$$\frac{\Delta}{S}\left(\frac{2S-(x+z)}{(S-x)(S-z)}\right)=\frac{y}{S}$$

$$\Rightarrow \quad \frac{\Delta y}{S(S-x)(S-z)}=\frac{y}{S}$$

$$\Rightarrow \quad \Delta^2=(S-x)^2\,(S-z)^2$$

$$\Rightarrow \quad S(S-y)=(S-x)\,(S-z)$$

$$\Rightarrow \quad (x+y+z)\,(x+z-y)=(y+z-x)\,(x+y-z)$$

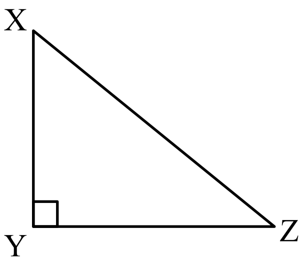
$$\Rightarrow \quad (x+z)^2-y^2=y^2-(z-x)^2$$

$$\Rightarrow \quad (x+z)^2+(x-z)^2=2y^2$$

$$\Rightarrow \quad x^2+z^2=y^2 \quad \Rightarrow \angle Y=\frac{\pi}{2}$$

$$\Rightarrow \quad \angle Y=\angle X+\angle Z$$

$$\tan\frac{x}{2}=\frac{\Delta}{S(S-x)}$$



$$\tan\frac{x}{2}=\frac{\frac{1}{2}xz}{\frac{(y+z)^2-x^2}{4}}$$

$$\tan\frac{x}{2}=\frac{2xz}{y^2+z^2+2yz-x^2}$$

$$\tan\frac{x}{2}=\frac{2xz}{2z^2+2yz} \qquad (\text{using } y^2=x^2+z^2)$$

$$\tan\frac{x}{2}=\frac{x}{y+z}$$

11. Let L_1 and L_2 be the following straight line.

$$\mathbf{L_1} : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} \text{ and } L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$$

Suppose the straight line

$$L : \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$$

lies in the plane containing L_1 and L_2, and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE?

(A) $\alpha - \gamma = 3$ (B) $l + m = 2$ (C) $\alpha - \gamma = 1$ (D) $l + m = 0$

Ans. (A,B)

Sol. Point of intersection of L_1 & L_2 is (1, 0, 1)

Line L passes through (1, 0, 1)

$$\frac{1-\alpha}{\ell} = -\frac{1}{m} = \frac{1-\gamma}{-2} \qquad ...(1)$$

acute angle bisector of L_1 & L_2

$$\vec{r} = \hat{i} + \hat{k} + \lambda\left(\frac{\hat{i} - \hat{j} + 3\hat{k} - 3\hat{i} - \hat{j} + \hat{k}}{\sqrt{11}}\right)$$

$$\vec{r} = \hat{i} + \hat{k} + t\left(\hat{i} + \hat{j} - 2\hat{k}\right)$$

$$\Rightarrow \quad \frac{\ell}{1} = \frac{m}{1} = \frac{-2}{-2} \quad \Rightarrow \quad \ell = m = 1$$

From (1) $\dfrac{1-\alpha}{1} = -1 \quad \Rightarrow \quad \alpha = 2$

$$\& \ \frac{1-\gamma}{-2} = -1 \quad \Rightarrow \quad \gamma = -1$$

12. Which of the following inequalities is/are TRUE?

(A) $\displaystyle\int_0^1 x\cos x\,dx \ge \frac{3}{8}$ (B) $\displaystyle\int_0^1 x\sin x\,dx \ge \frac{3}{10}$

(C) $\displaystyle\int_0^1 x^2\cos x\,dx \ge \frac{1}{2}$ (D) $\displaystyle\int_0^1 x^2\sin x\,dx \ge \frac{2}{9}$

Ans. (A,B,D)

Sol. (A) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$

$\cos x \ \geq \ 1 - \frac{x^2}{2}$

$\int_0^1 x\cos x \geq \int_0^1 x\left(1 - \frac{x^2}{2}\right) = \frac{1}{2} - \frac{1}{8}$

$\int_0^1 x\cos x \geq \frac{3}{8}$ (True)

(B) $\sin x \geq x - \frac{x^3}{6}$

$\int_0^1 x\sin x \geq \int_0^1 x\left(x - \frac{x^3}{6}\right)dx$

$\int_0^1 x\sin x \geq \frac{1}{3} - \frac{1}{30} \Rightarrow \int_0^1 x\sin x\ dx \geq \frac{3}{8}$ (True)

(D) $\int_0^1 x^2 \sin x\ dx \geq \int_0^1 x^2\left(x - \frac{x^3}{6}\right)dx$

$\int_0^1 x^2 \sin x\ dx \geq \frac{1}{4} - \frac{1}{36}$

$\int_0^1 x^2 \sin x\ dx \geq \frac{2}{9}$ (True)

(C) $\cos x < 1$

$x^2 \cos x < x^2$

$\int_0^1 x^2 \cos x\ dx < \int_0^1 x^2 dx$

$\int_0^1 x^2 \cos x\ dx < \frac{1}{3}$

So option 'C' is incorrect.

SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks : +4 If ONLY the correct numerical value is entered;

Zero Marks : 0 In all other cases.

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13. Let m be the minimum possible value of $\log_3\left(3^{y_1}+3^{y_2}+3^{y_3}\right)$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is ______.

Ans. (8.00)

Sol. $\dfrac{3^{y_1}+3^{y_2}+3^{y_3}}{3} \ge \left[3^{(y_1+y_2+y_3)}\right]^{\frac{1}{3}}$

$\Rightarrow\ 3^{y_1}+3^{y_2}+3^{y_3} \ge 3^4$

$\Rightarrow\ \log_3(3^{y_1}+3^{y_2}+3^{y_3}) \ge 4$

$\Rightarrow\ m = 4$

Also, $\dfrac{x_1+x_2+x_3}{3} \ge \sqrt[3]{x_1x_2x_3}$

$\Rightarrow\ x_1x_2x_3 \le 27$

$\Rightarrow\ \log_3 x_1 + \log_3 x_2 + \log_3 x_3 \le 3$

$\Rightarrow\ M = 3$

Thus, $\log_2(m^3) + \log_3(M^2) = 6 + 2$

$= 8$

14. Let a_1, a_2, a_3, ….. be a sequence of positive integers in arithmetic progression with common difference 2. Also, let b_1, b_2, b_3, ….. be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$$2(a_1 + a_2 + \ldots + a_n) = b_1 + b_2 + \ldots + b_n$$

holds for some positive integer n, is ______

Ans. (1.00)

Sol. Given $2(a_1 + a_2 + ... + a_n) = b_1 + b_2 + ... + b_n$

$\Rightarrow \quad 2 \times \frac{n}{2}(2c + (n-2)x_2) = c\left(\frac{2^n - 1}{2-1}\right)$

$\Rightarrow \quad 2n^2 - 2n = c(2^n - 1 - 2n)$

$\Rightarrow \quad c = \frac{2n^2 - 2n}{2^n - 1 - 2n} \in \mathbb{N}$

So, $\quad 2n^2 - 2n \geq 2^n - 1 - 2n$

$\Rightarrow \quad 2n^2 + 1 \geq 2^n \Rightarrow n < 7$

$\Rightarrow \quad$ n can be 1,2,3,....,

Checking c against these values of n

we get c = 12 (when n = 3)

Hence number of such c = 1

15. Let $f : [0, 2] \to \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is _____

Ans. (1.00)

Sol. Let $\pi x - \frac{\pi}{4} = \theta \in \left[\frac{-\pi}{4}, \frac{7\pi}{4}\right]$

So, $\left(3 - \sin\left(\frac{\pi}{2} + 2\theta\right)\right)\sin\theta \geq \sin(\pi + 3\theta)$

$\Rightarrow \quad (3 - \cos2\theta)\sin\theta \geq -\sin3\theta$

$\Rightarrow \quad \sin\theta[3 - 4\sin^2\theta + 3 - \cos2\theta] \geq 0$

$\Rightarrow \quad \sin\theta(6 - 2(1 - \cos2\theta) - \cos2\theta) \geq 0$

$\Rightarrow \quad \sin\theta(4 + \cos2\theta) \geq 0$

$\Rightarrow \quad \sin\theta \geq 0$

$\Rightarrow \quad \theta \in [0, \pi] \Rightarrow 0 \leq \pi x - \frac{\pi}{4} \leq \pi$

$\Rightarrow \quad x \in \left[\frac{1}{4}, \frac{5}{4}\right]$

$\Rightarrow \quad \beta - \alpha = 1$

16. In a triangle PQR, let $\vec{a} = \overrightarrow{QR}, \vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If

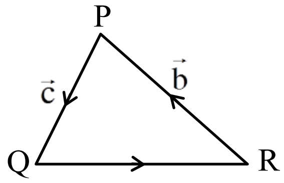
$|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\frac{\vec{a}.(\vec{c} - \vec{b})}{\vec{c}.(\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$, then the value of $|\vec{a} \times \vec{b}|^2$ is ______

Ans. (108.00)

Sol. We have $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \quad \vec{c}=-\vec{a}-\vec{b}$

Now, $\dfrac{\vec{a}.(-\vec{a}-2\vec{b})}{(-\vec{a}-\vec{b}).(\vec{a}-\vec{b})}=\dfrac{3}{7}$



$\Rightarrow \quad \dfrac{9+2\vec{a}.\vec{b}}{9-16}=\dfrac{3}{7}$

$\Rightarrow \quad \vec{a}.\vec{b}=-6$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2=a^2b^2-(\vec{a}.\vec{b})^2=9\times 16-36=108$

17. For a polynomial g(x) with real coefficient, let m_g denote the number of distinct real roots of g(x). Suppose S is the set of polynomials with real coefficient defined by

$S=\{(x^2-1)^2(a_0+a_1x+a_2x^2+a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$.

For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'}+m_{f''})$, where $f \in S$, is _______

Ans. (5.00)

Sol. $f(x)=(x^2-1)^2\,h(x)$; $h(x)=a_0+a_1x+a_2x^2+a_3x^3$

Now, $f(1)=f(-1)=0$

$\Rightarrow \quad f'(\alpha)=0,\ \alpha\in(-1,1)$ [Rolle's Theorem]

Also, $f'(1)=f'(-1)=0 \Rightarrow f'(x)=0$ has atleast 3 root, $-1,\alpha,1$ with $-1<\alpha<1$

$\Rightarrow \quad f''(x)=0$ will have at leeast 2 root, say β,γ such that

$-1<\beta<\alpha<\gamma<1$ [Rolle's Theorem]

So, $\min(m_{f''})=2$

and we find $(m_{f'}+m_{f''})=5$ for $f(x)=(x^2-1)^2$.

Thus, Ans. 5

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x\to 0^+}\frac{(1-x)^{\frac{1}{x}}-e^{-1}}{x^a}$$

is equal to a nonzero real number, is ______.

Ans. (1.00)

Sol. $\lim\limits_{x \to 0^+} \dfrac{e^{\left(\frac{\ell n(1-x)}{x}\right)} - \frac{1}{e}}{x^a}$

$$= \lim_{x \to 0^+} \frac{1}{e} \; \frac{e^{\left(1+\frac{\ell n(1-x)}{x}\right)} - 1}{x^a}$$

$$= \frac{1}{e} \lim_{x \to 0^+} \frac{1 + \dfrac{\ell n(1-x)}{x}}{x^a}$$

$$= \frac{1}{e} \lim_{x \to 0^+} \frac{\ell n(1-x) + x}{x^{(a+1)}}$$

$$= \frac{1}{e} \lim_{x \to 0^+} \frac{\left(-x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - ....\right) + x}{x^{a+1}}$$

Thus, $a = 1$