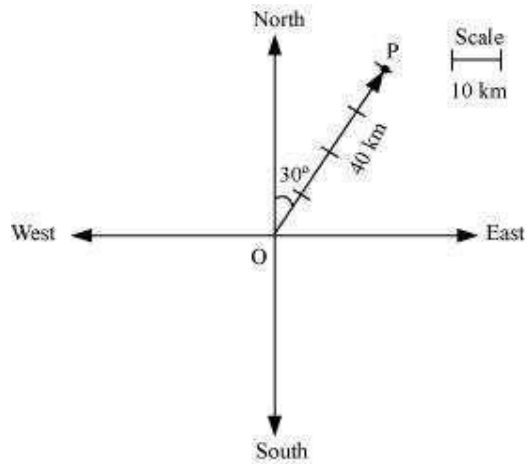
Exercise 10.1

Question 1:

Represent graphically a displacement of 40 km, 30° east of north.

Answer



Here, vector $\overrightarrow{OP}$ represents the displacement of 40 km, 30° East of North.

Question 2:

Classify the following measures as scalars and vectors.

(i) 10 kg (ii) 2 metres north-west (iii) 40°

(iv) 40 watt (v) 10^{-19} coulomb (vi) 20 m/s^2

Answer

(i) 10 kg is a scalar quantity because it involves only magnitude.

(ii) 2 meters north-west is a vector quantity as it involves both magnitude and direction.

(iii) 40° is a scalar quantity as it involves only magnitude.

(iv) 40 watts is a scalar quantity as it involves only magnitude.

(v) 10^{-19} coulomb is a scalar quantity as it involves only magnitude.

(vi) 20 m/s^2 is a vector quantity as it involves magnitude as well as direction.

Question 3:

Classify the following as scalar and vector quantities.

(i) time period (ii) distance (iii) force

(iv) velocity (v) work
done Answer

(i) Time period is a scalar quantity as it involves only magnitude.

(ii) Distance is a scalar quantity as it involves only magnitude.

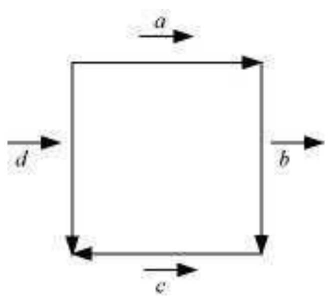
(iii) Force is a vector quantity as it involves both magnitude and direction.

(iv) Velocity is a vector quantity as it involves both magnitude as well as direction.

(v) Work done is a scalar quantity as it involves only magnitude.

Question 4:

In Figure, identify the following vectors.



(i) Coinitial (ii) Equal (iii) Collinear but not
equal Answer

(i) Vectors $\vec{a}$ and $\vec{d}$ are coinitial because they have the same initial point.

(ii) Vectors $\vec{b}$ and $\vec{d}$ are equal because they have the same magnitude and direction.

(iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal. This is because although they are parallel, their directions are not the same.

Question 5:

Answer the following as true or false.

(i) $\vec{a}$ and $-\vec{a}$ are collinear.

(ii) Two collinear vectors are always equal in magnitude.

(iii) Two vectors having same magnitude are collinear.

(iv) Two collinear vectors having the same magnitude are
equal. Answer

(i) True.

Vectors $\vec{a}$ and $-\vec{a}$ are parallel to the same line. (ii) False.

Collinear vectors are those vectors that are parallel to the same line.

(iii) False.

Exercise 10.2

Question 1:

Compute the magnitude of the following vectors:

$\vec{a}=\hat{i}+\hat{j}+\hat{k};\quad \vec{b}=2\hat{i}-7\hat{j}-3\hat{k};\qquad \vec{c}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$

Answer

The given vectors are:

$\vec{a}=\hat{i}+\hat{j}+\hat{k};\quad \vec{b}=2\hat{i}-7\hat{j}-3\hat{k};\qquad \vec{c}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$

$$|\vec{a}|=\sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$$

$$\begin{aligned}|\vec{b}|&=\sqrt{(2)^2+(-7)^2+(-3)^2}\\&=\sqrt{4+49+9}\\&=\sqrt{62}\end{aligned}$$

$$\begin{aligned}|\vec{c}|&=\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(-\frac{1}{\sqrt{3}}\right)^2}\\&=\sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=1\end{aligned}$$

Question 2:

Write two different vectors having same magnitude.

Answer

Consider $\vec{a}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)$ and $\vec{b}=\left(2\hat{i}+\hat{j}-3\hat{k}\right)$.

It can be observed that $|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$ and

$|\vec{b}|=\sqrt{2^2+1^2+(-3)^2}=\sqrt{4+1+9}=\sqrt{14}$.

Hence, $\vec{a}$ and $\vec{b}$ are two different vectors having the same magnitude. The vectors are different because they have different directions.

Question 3:

Write two different vectors having same direction.

Answer

Consider $\vec{p} = \left(\hat{i} + \hat{j} + \hat{k}\right)$ and $\vec{q} = \left(2\hat{i} + 2\hat{j} + 2\hat{k}\right)$.

The direction cosines of $\vec{p}$ are given by,

$$l = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}},\ m = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}},\ \text{and } n = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}.$$

The direction cosines of $\vec{q}$ are given by

$$l = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},\ m = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},$$

$$\text{and } n = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

The direction cosines of $\vec{p}$ and $\vec{q}$ are the same. Hence, the two vectors have the same direction.

Question 4:

Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal Answer

The two vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ will be equal if their corresponding components are equal.

Hence, the required values of x and y are 2 and 3 respectively.

Question 5:

Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7).

Answer

The vector with the initial point P (2, 1) and terminal point Q (–5, 7) can be given by,

$$\overrightarrow{PQ} = (-5-2)\hat{i} + (7-1)\hat{j}$$

$$\Rightarrow \overrightarrow{PQ} = -7\hat{i} + 6\hat{j}$$

Hence, the required scalar components are –7 and 6 while the vector components are $-7\hat{i}$ and $6\hat{j}$.

Question 6:

Find the sum of the vectors $\vec{a}=\hat{i}-2\hat{j}+\hat{k},\ \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$. Answer

The given vectors are $\vec{a}=\hat{i}-2\hat{j}+\hat{k},\ \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$.

$$\begin{aligned}\therefore \vec{a}+\vec{b}+\vec{c}&=(1-2+1)\hat{i}+(-2+4-6)\hat{j}+(1+5-7)\hat{k}\\&=0\cdot\hat{i}-4\hat{j}-1\cdot\hat{k}\\&=-4\hat{j}-\hat{k}\end{aligned}$$

Question 7:

Find the unit vector in the direction of the vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$.

Answer

The unit vector $\hat{a}$ in the direction of vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ is given by $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$.

$$|\vec{a}|=\sqrt{1^2+1^2+2^2}=\sqrt{1+1+4}=\sqrt{6}$$

$$\therefore \hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}=\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$$

Question 8:

Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.

Answer

The given points are P (1, 2, 3) and Q (4, 5, 6).

$$\therefore \overrightarrow{PQ}=(4-1)\hat{i}+(5-2)\hat{j}+(6-3)\hat{k}=3\hat{i}+3\hat{j}+3\hat{k}$$

$$\left|\overrightarrow{PQ}\right|=\sqrt{3^2+3^2+3^2}=\sqrt{9+9+9}=\sqrt{27}=3\sqrt{3}$$

Hence, the unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|}=\frac{3\hat{i}+3\hat{j}+3\hat{k}}{3\sqrt{3}}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$$

.

Question 9:

For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$

Answer

The given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$.

$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

$\therefore \vec{a} + \vec{b} = (2-1)\hat{i} + (-1+1)\hat{j} + (2-1)\hat{k} = 1\hat{i} + 0\hat{j} + 1\hat{k} = \hat{i} + \hat{k}$

$\left|\vec{a} + \vec{b}\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Hence, the unit vector in the direction of $\left(\vec{a} + \vec{b}\right)$ is

$$\frac{\left(\vec{a} + \vec{b}\right)}{\left|\vec{a} + \vec{b}\right|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

.

Question 10:

Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units. Answer

Let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$.

$\therefore |\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25 + 1 + 4} = \sqrt{30}$

$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}$

Hence, the vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units is given by,

$$8\hat{a} = 8\left(\frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}\right) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$$

$$= 8\left(\frac{5\vec{i} - \vec{j} + 2\vec{k}}{\sqrt{30}}\right)$$

$$= \frac{40}{\sqrt{30}}\vec{i} - \frac{8}{\sqrt{30}}\vec{j} + \frac{16}{\sqrt{30}}\vec{k}$$

Question 11:

Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

Answer

Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$.

It is observed that $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k} = -2\left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) = -2\vec{a}$

$\therefore \vec{b} = \lambda\vec{a}$

where,

$\lambda = -2$

Hence, the given vectors are collinear.

Question 12:

Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$

Answer

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

$\therefore |\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

Hence, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

Question 13:

Find the direction cosines of the vector joining the points A (1, 2, –3) and B (–1, –2, 1) directed from A to B.

Answer

The given points are A (1, 2, –3) and B (–1, –2, 1).

$$\therefore \overrightarrow{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k}$$
$$\Rightarrow \overrightarrow{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$$
$$\therefore \left|\overrightarrow{AB}\right| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4+16+16} = \sqrt{36} = 6$$

Hence, the direction cosines of $\overrightarrow{AB}$ are $\left(-\frac{2}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$.

Question 14:

Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY, and OZ. Answer

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Then,

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Therefore, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Now, let α, β, and γbe the angles formed by $\vec{a}$ with the positive directions of x, y, and z axes.

Then, we have $\cos\alpha = \frac{1}{\sqrt{3}}, \cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{1}{\sqrt{3}}$.

Hence, the given vector is equally inclined to axes OX, OY, and OZ.

Question 15:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ration 2:1

(i) internally

(ii) externally

Answer

The position vector of point R dividing the line segment joining two points P and Q in the ratio m: n is given by:

i. Internally:

$$\frac{m\vec{b}+n\vec{a}}{m+n}$$

ii. Externally:

$$\frac{m\vec{b}-n\vec{a}}{m-n}$$

Position vectors of P and Q are given as:

$\overrightarrow{OP}=\hat{i}+2\hat{j}-\hat{k}$ and $\overrightarrow{OQ}=-\hat{i}+\hat{j}+\hat{k}$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2:1 is given by,

$$\overrightarrow{OR}=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1}=\frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-\hat{k}\right)}{3}$$

$$=\frac{-\hat{i}+4\hat{j}+\hat{k}}{3}=-\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2:1 is given by,

$$\overrightarrow{OR}=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)-1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2-1}=\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-\hat{k}\right)$$

$$=-3\hat{i}+3\hat{k}$$

Question 16:

Find the position vector of the mid point of the vector joining the points P (2, 3, 4) and Q (4, 1, – 2).

Answer

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, – 2) is given by,

$$\overrightarrow{OR}=\frac{\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\left(4\hat{i}+\hat{j}-2\hat{k}\right)}{2}=\frac{(2+4)\hat{i}+(3+1)\hat{j}+(4-2)\hat{k}}{2}$$

$$=\frac{6\hat{i}+4\hat{j}+2\hat{k}}{2}=3\hat{i}+2\hat{j}+\hat{k}$$

Question 17:

Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$,

$\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right angled triangle. Answer

Position vectors of points A, B, and C are respectively given as:

$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

$$\therefore \overrightarrow{AB} = \vec{b} - \vec{a} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\overrightarrow{BC} = \vec{c} - \vec{b} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{CA} = \vec{a} - \vec{c} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\therefore \left|\overrightarrow{AB}\right|^2 = (-1)^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35$$

$$\left|\overrightarrow{BC}\right|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41$$

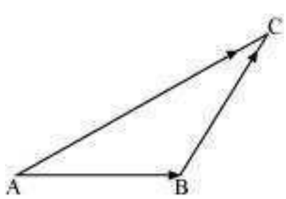
$$\left|\overrightarrow{CA}\right|^2 = 2^2 + (-1)^2 + 1^2 = 4 + 1 + 1 = 6$$

$$\therefore \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 = 36 + 6 = 41 = \left|\overrightarrow{BC}\right|^2$$

Hence, ABC is a right-angled triangle.

Question 18:

In triangle ABC which of the following is not true:



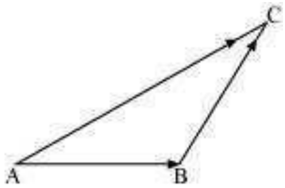
A. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

B. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

C. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

D. $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

Answer



On applying the triangle law of addition in the given triangle, we have:

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC} \quad ...(1)$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0} \quad ...(2)$

$\therefore$ The equation given in alternative A is true.

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

$\therefore$ The equation given in alternative B is true.

From equation (2), we have:

$\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$

$\therefore$ The equation given in alternative D is true.

Now, consider the equation given in alternative C:

$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=\vec{0}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA} \quad ...(3)$

From equations (1) and (3), we have:

$\overrightarrow{AC}=\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AC}=-\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AC}+\overrightarrow{AC}=\vec{0}$

$\Rightarrow 2\overrightarrow{AC}=\vec{0}$

$\Rightarrow \overrightarrow{AC}=\vec{0}$, which is not true.

Hence, the equation given in alternative C is incorrect.

The correct answer is C.

Question 19:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

A. $\vec{b} = \lambda\vec{a}$, for some scalar λ

B. $\vec{a} = \pm\vec{b}$

C. the respective components of $\vec{a}$ and $\vec{b}$ are proportional

D. both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes Answer

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel.
Therefore, we have:

$\vec{b} = \lambda\vec{a}$ (For some scalar λ)

If λ = ±1, then $\vec{a} = \pm\vec{b}$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{b} = \lambda\vec{a}.$$

$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$

$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

$$\Rightarrow b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$$

$$\Rightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Thus, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

However, vectors $\vec{a}$ and $\vec{b}$ can have different directions.
Hence, the statement given in D is incorrect.
The correct answer is D.

Exercise 10.3

Question 1:

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively

having $\vec{a}\cdot\vec{b}=\sqrt{6}$

. Answer

It is given that,

$|\vec{a}|=\sqrt{3},\ |\vec{b}|=2 \text{ and, } \vec{a}\cdot\vec{b}=\sqrt{6}$

Now, we know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

$\therefore \sqrt{6}=\sqrt{3}\times 2\times\cos\theta$

$\Rightarrow \cos\theta=\dfrac{\sqrt{6}}{\sqrt{3}\times 2}$

$\Rightarrow \cos\theta=\dfrac{1}{\sqrt{2}}$

$\Rightarrow \theta=\dfrac{\pi}{4}$

Hence, the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{4}$.

Question 2:

Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k} \text{ and } 3\hat{i}-2\hat{j}+\hat{k}$

Answer

The given vectors are $\vec{a}=\hat{i}-2\hat{j}+3\hat{k} \text{ and } \vec{b}=3\hat{i}-2\hat{j}+\hat{k}$.

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\begin{aligned}\text{Now, } \vec{a}\cdot\vec{b} &= (\hat{i} - 2\hat{j} + 3\hat{k})(3\hat{i} - 2\hat{j} + \hat{k}) \\ &= 1.3 + (-2)(-2) + 3.1 \\ &= 3 + 4 + 3 \\ &= 10\end{aligned}$$

Also, we know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore 10 = \sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Question 3:

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.
Answer

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{1+1}}\{1.1 + (-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$$

Hence, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

Question 4:

Find the projection of the vector $\hat{i} + 3\hat{i} + 7\hat{k}$ on the vector $7\hat{i} - \hat{i} + 8\hat{k}$. Answer

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\hat{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})=\frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{1(7)+3(-1)+7(8)\}=\frac{7-3+56}{\sqrt{49+1+64}}=\frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}),\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}),\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

Answer

$$\text{Let } \vec{a}=\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k},$$

$$\vec{b}=\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k},$$

$$\vec{c}=\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})=\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}.$$

$$|\vec{a}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$

$$|\vec{b}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}}=1$$

$$|\vec{c}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a}\cdot\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(\frac{-6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$

$$\vec{b}\cdot\vec{c}=\frac{3}{7}\times\frac{6}{7}+\left(\frac{-6}{7}\right)\times\frac{2}{7}+\frac{2}{7}\times\left(\frac{-3}{7}\right)=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$

$$\vec{c}\cdot\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\left(\frac{-3}{7}\right)\times\frac{6}{7}=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$$

Hence, the given three vectors are mutually perpendicular to each other.

Question 6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$.

Answer

$$(\vec{a}\cdot\vec{b}).(\vec{a}-\vec{b})=8$$

$$\Rightarrow \vec{a}\cdot\vec{a}-\vec{a}.\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=8$$

$$\Rightarrow |\vec{a}|^2-|\vec{b}|^2=8$$

$$\Rightarrow (8|\vec{b}|)^2-|\vec{b}|^2=8 \qquad \left[|\vec{a}|=8|\vec{b}|\right]$$

$$\Rightarrow 64|\vec{b}|^2-|\vec{b}|^2=8$$

$$\Rightarrow 63|\vec{b}|^2=8$$

$$\Rightarrow |\vec{b}|^2=\frac{8}{63}$$

$$\Rightarrow |\vec{b}|=\sqrt{\frac{8}{63}} \qquad [\text{Magnitude of a vector is non-negative}]$$

$$\Rightarrow |\vec{b}|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}|=8|\vec{b}|=\frac{8\times 2\sqrt{2}}{3\sqrt{7}}=\frac{16\sqrt{2}}{3\sqrt{7}}$$

Question 7:

Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

Answer

$$(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$$

$$=3\vec{a}\cdot 2\vec{a}+3\vec{a}\cdot 7\vec{b}-5\vec{b}\cdot 2\vec{a}-5\vec{b}\cdot 7\vec{b}$$

$$=6\vec{a}\cdot\vec{a}+21\vec{a}\cdot\vec{b}-10\vec{a}\cdot\vec{b}-35\vec{b}\cdot\vec{b}$$

$$=6|\vec{a}|^2+11\vec{a}\cdot\vec{b}-35|\vec{b}|^2$$

Question 8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that

the angle between them is 60° and their scalar product is $\frac{1}{2}$

Answer

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

It is given that $|\vec{a}| = |\vec{b}|$, $\vec{a} \cdot \vec{b} = \frac{1}{2}$, and $\theta = 60°$. ...(1)

We know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \frac{1}{2} = |\vec{a}||\vec{a}|\cos 60° \qquad [\text{Using } (1)]$$

$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2 = 1$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

Question 9:

Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$.

Answer

$$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$$

$$\Rightarrow \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12 \qquad [|\vec{a}| = 1 \text{ as } \vec{a} \text{ is a unit vector}]$$

$$\Rightarrow |\vec{x}|^2 = 13$$

$$\therefore |\vec{x}| = \sqrt{13}$$

Question 10:

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k},\ \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Answer

The given vectors are $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k},\ \vec{b} = -\hat{i} + 2\hat{j} + \hat{k},$ and $\vec{c} = 3\hat{i} + \hat{j}.$

Now,

$\vec{a} + \lambda\vec{b} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-\hat{i} + 2\hat{j} + \hat{k}\right) = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$

If $\left(\vec{a} + \lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then

$\left(\vec{a} + \lambda\vec{b}\right)\cdot\vec{c} = 0.$

$\Rightarrow \left[(2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}\right]\cdot\left(3\hat{i} + \hat{j}\right) = 0$

$\Rightarrow (2-\lambda)3 + (2+2\lambda)1 + (3+\lambda)0 = 0$

$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$

$\Rightarrow -\lambda + 8 = 0$

$\Rightarrow \lambda = 8$

Hence, the required value of λ is 8.

Question 11:

Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$ Answer

$\left(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\right)\cdot\left(|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\right)$

$= |\vec{a}|^2\,\vec{b}\cdot\vec{b} - |\vec{a}||\vec{b}|\vec{b}\cdot\vec{a} + |\vec{b}||\vec{a}|\vec{a}\cdot\vec{b} - |\vec{b}|^2\,\vec{a}\cdot\vec{a}$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{b}|^2|\vec{a}|^2$

$= 0$

Hence, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ are perpendicular to each other.

Question 12:

If $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

Answer

It is given that $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$.

Now,

$\vec{a}\cdot\vec{a}=0 \Rightarrow |\vec{a}|^2=0 \Rightarrow |\vec{a}|=0$

$\therefore \vec{a}$ is a zero vector.

Hence, vector $\vec{b}$ satisfying $\vec{a}\cdot\vec{b}=0$ can be any vector.

Question 14:

If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

Answer

Consider $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$.

Then,

$\vec{a}\cdot\vec{b}=2.3+4.3+3(-6)=6+12-18=0$

We now observe that:

$|\vec{a}|=\sqrt{2^2+4^2+3^2}=\sqrt{29}$

$\therefore \vec{a}\neq\vec{0}$

$|\vec{b}|=\sqrt{3^2+3^2+(-6)^2}=\sqrt{54}$

$\therefore \vec{b}\neq\vec{0}$

Hence, the converse of the given statement need not be true.

Question 15:

If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Answer

The vertices of ΔABC are given as A (1, 2, 3), B (–1, 0, 0), and C (0, 1, 2).

Also, it is given that ∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \{1-(-1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = \{0-(-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \overrightarrow{BA} \cdot \overrightarrow{BC} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) \cdot \left(\hat{i} + \hat{j} + 2\hat{k}\right) = 2\times1 + 2\times1 + 3\times2 = 2+2+6 = 10$$

$$\left|\overrightarrow{BA}\right| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1+1+2^2} = \sqrt{6}$$

Now, it is known that:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos(\angle ABC).$$

$$\therefore 10 = \sqrt{17} \times \sqrt{6}\cos(\angle ABC)$$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17}\times\sqrt{6}}$$

$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Question 16:

Show that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, –1) are collinear. Answer

The given points are A (1, 2, 7), B (2, 6, 3), and C (3, 10, –1).

$$\therefore \overrightarrow{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$$

$$\therefore \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Hence, the given points A, B, and C are collinear.

Question 17:

Show that the vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Answer

Let vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ be position vectors of points A, B, and C respectively.

i.e., $\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k},\ \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k}$ and $\overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

Now, vectors $\overrightarrow{AB}, \overrightarrow{BC}$, and $\overrightarrow{AC}$ represent the sides of ΔABC.

i.e., $\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k},\ \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k}$, and $\overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

$\therefore \overrightarrow{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$

$\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{AC}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$

$|\overrightarrow{AB}|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$

$|\overrightarrow{BC}|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$

$|\overrightarrow{AC}|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$

$\therefore |\overrightarrow{BC}|^2+|\overrightarrow{AC}|^2=6+35=41=|\overrightarrow{AB}|^2$

Hence, ΔABC is a right-angled triangle.

Question 18:

If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then λ$\vec{a}$ is unit vector if

(A) λ = 1 (B) λ = −1 (C) $a=|\lambda|$ (D) $a=\frac{1}{|\lambda|}$

Answer

Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}|=1$.

Now,

$|\lambda\vec{a}| = 1$

$\Rightarrow |\lambda||\vec{a}| = 1$

$\Rightarrow |\vec{a}| = \frac{1}{|\lambda|}$ $[\lambda \neq 0]$

$\Rightarrow a = \frac{1}{|\lambda|}$ $[|\vec{a}| = a]$

Hence, vector $\lambda\vec{a}$ is a unit vector if $a = \frac{1}{|\lambda|}$.

The correct answer is D.

Exercise 10.4

Question 1:

Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$.

Answer

We have,

$\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)=19\hat{j}+19\hat{k}$$

$$\therefore\left|\vec{a}\times\vec{b}\right|=\sqrt{(19)^2+(19)^2}=\sqrt{2\times(19)^2}=19\sqrt{2}$$

Question 2:

Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Answer

We have,

$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$$\therefore\vec{a}+\vec{b}=4\hat{i}+4\hat{j},\ \vec{a}-\vec{b}=2\hat{i}+4\hat{k}$$

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\therefore\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\sqrt{16^2+(-16)^2+(-8)^2}$$

$$=\sqrt{2^2\times8^2+2^2\times8^2+8^2}$$

$$=8\sqrt{2^2+2^2+1}=8\sqrt{9}=8\times3=24$$

Hence, the unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by the relation,

$$=\pm\frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|}=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$$

$$=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}=\pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}$$

Question 3:

If a unit vector $\vec{a}$ makes an $\frac{\pi}{3}$ with $\hat{i}, \frac{\pi}{4}$ angles with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the compounds of $\vec{a}$.

Answer

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$$

Since $\vec{a}$ is a unit vector, $|\vec{a}|=1$.

Also, it is given that $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$ $\frac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$.

Then, we have:

$$\cos\frac{\pi}{3}=\frac{a_1}{|\vec{a}|}$$

$$\Rightarrow\frac{1}{2}=a_1 \qquad \left[|\vec{a}|=1\right]$$

$$\cos\frac{\pi}{4}=\frac{a_2}{|\vec{a}|}$$

$$\Rightarrow\frac{1}{\sqrt{2}}=a_2 \qquad \left[|\vec{a}|=1\right]$$

$$\text{Also, }\cos\theta=\frac{a_3}{|\vec{a}|}.$$

$$\Rightarrow a_3=\cos\theta$$

Now,

$|a| = 1$

$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$

$\Rightarrow \frac{3}{4} + \cos^2\theta = 1$

$\Rightarrow \cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$

Hence, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Question 4:

Show that

$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$

Answer

$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right)$

$= \left(\vec{a} - \vec{b}\right) \times \vec{a} + \left(\vec{a} - \vec{b}\right) \times \vec{b}$ [By distributivity of vector product over addition]

$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$ [Again, by distributivity of vector product over addition]

$= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$

$= 2\vec{a} \times \vec{b}$

Question 5:

Find λ and μ if $\left(2\hat{i} + 6\hat{j} + 27\hat{k}\right) \times \left(\hat{i} + \lambda\hat{j} + \mu\hat{k}\right) = \vec{0}$.

Answer

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$$

$$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 27\\ 1 & \lambda & \mu\end{vmatrix}=0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow\hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have:

$6\mu-27\lambda=0$

$2\mu-27=0$

$2\lambda-6=0$

Now,

$2\lambda-6=0\Rightarrow\lambda=3$

$2\mu-27=0\Rightarrow\mu=\dfrac{27}{2}$

Hence, $\lambda=3$ and $\mu=\dfrac{27}{2}$.

Question 6:

Given that $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$? Answer

$\vec{a}\cdot\vec{b}=0$

Then,

(i) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, or $\vec{a}\perp\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

$\vec{a}\times\vec{b}=0$

(ii) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, or $\vec{a}\parallel\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Hence, $|\vec{a}|=0$ or $|\vec{b}|=0$.

Question 7:

Let the vectors $\vec{a},\ \vec{b},\ \vec{c}$ given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Then show that $=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Answer

We have,

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$$

$$\left(\vec{b}+\vec{c}\right)=\left(b_1+c_1\right)\hat{i}+\left(b_2+c_2\right)\hat{j}+\left(b_3+c_3\right)\hat{k}$$

$$\text{Now, } \vec{a}\times\left(\vec{b}+\vec{c}\right)\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$$

$$=\hat{i}\left[a_2\left(b_3+c_3\right)-a_3\left(b_2+c_2\right)\right]-\hat{j}\left[a_1\left(b_3+c_3\right)-a_3\left(b_1+c_1\right)\right]+\hat{k}\left[a_1\left(b_2+c_2\right)-a_2\left(b_1+c_1\right)\right]$$

$$=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3-a_1c_3+a_3b_1+a_3c_1\right]+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]\quad \dots(1)$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[b_1a_3-a_1b_3\right]+\hat{k}\left[a_1b_2-a_2b_1\right]\qquad(2)$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_1c_2-a_2c_1\right]\qquad(3)$$

On adding (2) and (3), we get:

$$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[b_1a_3+a_3c_1-a_1b_3-a_1c_3\right]$$
$$+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]\qquad(4)$$

Now, from (1) and (4), we have:

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence, the given result is proved.

Question 8:

If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$. Is the converse true? Justify your answer with an example.

Answer

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$.

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k},\ \vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$.

Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}=\hat{i}(24-24)-\hat{j}(16-16)+\hat{k}(12-12)=0\hat{i}+0\hat{j}+0\hat{k}=\vec{0}$$

It can now be observed that:

$|\vec{a}|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$

$\therefore \vec{a}\neq\vec{0}$

$|\vec{b}|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$

$\therefore \vec{b}\neq\vec{0}$

Hence, the converse of the given statement need not be true.

Question 9:

Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5). Answer

The vertices of triangle ABC are given as A (1, 1, 2), B (2, 3, 5), and C (1, 5, 5).

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of ΔABC are given as:

$\overrightarrow{AB}=(2-1)\hat{i}+(3-1)\hat{j}+(5-2)\hat{k}=\hat{i}+2\hat{j}+3\hat{k}$

$\overrightarrow{BC}=(1-2)\hat{i}+(5-3)\hat{j}+(5-5)\hat{k}=-\hat{i}+2\hat{j}$

Area of ΔABC $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Answer

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a} \times \vec{b}\right|$. Adjacent sides are given as:

$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{20^2 + 5^2 + 5^2} = \sqrt{400+25+25} = 15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units.

Question 11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Answer

It is given that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$.

We know that $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}$ is a unit vector if $|\vec{a}\times\vec{b}| = 1$.

$|\vec{a}\times\vec{b}| = 1$

$\Rightarrow \left||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\right| = 1$

$\Rightarrow |\vec{a}||\vec{b}||\sin\theta| = 1$

$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta = 1$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}$

Hence, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

The correct answer is B.

Question 12:

Area of a rectangle having vertices A, B, C, and D with position vectors

$-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D)

4 Answer

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overrightarrow{OA} = -\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k},\ \overrightarrow{OB} = \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k},\ \overrightarrow{OC} = \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k},\ \overrightarrow{OD} = -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as:

$$\overrightarrow{AB} = (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = 2\hat{i}$$

$$\overrightarrow{BC} = (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = -\hat{j}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \hat{k}(-2) = -2\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-2)^2} = 2$$

Now, it is known that the area of a parallelogram whose adjacent sides are

$\vec{a}$ and $\vec{b}$ is $\left|\vec{a} \times \vec{b}\right|$.

Hence, the area of the given rectangle is
$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = 2$ square units. The correct answer is C.

Miscellaneous Solutions

Question 1:

Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

Answer

If $\vec{r}$ is a unit vector in the XY-plane, then $\vec{r} = \cos\theta\hat{i} + \sin\theta\hat{j}.$

Here, θ is the angle made by the unit vector with the positive direction of the x-axis. Therefore, for θ = 30°:

$$\vec{r} = \cos 30°\hat{i} + \sin 30°\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

Hence, the required unit vector is $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

Question 2:

Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Answer

The vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ can be obtained by,

$$\overrightarrow{PQ} = \text{Position vector of Q} - \text{Position vector of P}$$
$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

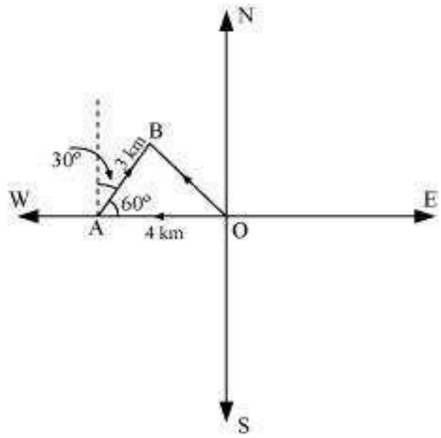
Hence, the scalar components and the magnitude of the vector joining the given points are respectively $\{(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)\}$ and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Question 3:

A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Answer

Let O and B be the initial and final positions of the girl respectively. Then, the girl's position can be shown as:



Now, we have:

$$\overrightarrow{OA} = -4\hat{i}$$

$$\overrightarrow{AB} = \hat{i}\left|\overrightarrow{AB}\right|\cos 60^\circ + \hat{j}\left|\overrightarrow{AB}\right|\sin 60^\circ$$

$$= \hat{i}3 \times \frac{1}{2} + \hat{j}3 \times \frac{\sqrt{3}}{2}$$

$$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

By the triangle law of vector addition, we have:

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$= \left(-4\hat{i}\right) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right)$$

$$= \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$= \left(\frac{-8+3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

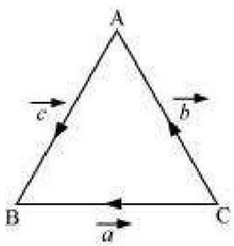
Hence, the girl's displacement from her initial point of departure is

$$\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}.$$

Question 4:

If $\vec{a}=\vec{b}+\vec{c}$, then is it true that $|\vec{a}|=|\vec{b}|+|\vec{c}|$? Justify your answer.
Answer

In ΔABC, let $\overrightarrow{CB}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$, and $\overrightarrow{AB}=\vec{c}$ (as shown in the following figure).



Now, by the triangle law of vector addition, we have $\vec{a}=\vec{b}+\vec{c}$.

It is clearly known that $|\vec{a}|$, $|\vec{b}|$, and $|\vec{c}|$ represent the sides of ΔABC.
Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$\therefore |\vec{a}|<|\vec{b}|+|\vec{c}|$

Hence, it is not true that $|\vec{a}|=|\vec{b}|+|\vec{c}|$.

Question 5:

Find the value of x for which $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector.
Answer

$x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector if $|x(\hat{i}+\hat{j}+\hat{k})|=1$.

Now,

$\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$

$\Rightarrow \sqrt{x^2+x^2+x^2}=1$

$\Rightarrow \sqrt{3x^2}=1$

$\Rightarrow \sqrt{3}\,x=1$

$\Rightarrow x=\pm\frac{1}{\sqrt{3}}$

Hence, the required value of x is $\pm\frac{1}{\sqrt{3}}$.

Question 6:

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.

Answer

We have,

$\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$.

Then,

$\vec{c}=\vec{a}+\vec{b}=(2+1)\hat{i}+(3-2)\hat{j}+(-1+1)\hat{k}=3\hat{i}+\hat{j}$

$\therefore |\vec{c}|=\sqrt{3^2+1^2}=\sqrt{9+1}=\sqrt{10}$

$\therefore \hat{c}=\frac{\vec{c}}{|\vec{c}|}=\frac{\left(3\hat{i}+\hat{j}\right)}{\sqrt{10}}$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$\pm 5\cdot\hat{c}=\pm 5\cdot\frac{1}{\sqrt{10}}\left(3\hat{i}+\hat{j}\right)=\pm\frac{3\sqrt{10}\hat{i}}{2}\pm\frac{\sqrt{10}}{2}\hat{j}.$

Question 7:

If $\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$

. Answer

We have,

$\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$

$$\begin{aligned}2\vec{a}-\vec{b}+3\vec{c}&=2\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(2\hat{i}-\hat{j}+3\hat{k}\right)+3\left(\hat{i}-2\hat{j}+\hat{k}\right)\\&=2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}\\&=3\hat{i}-3\hat{j}+2\hat{k}\end{aligned}$$

$$\left|2\vec{a}-\vec{b}+3\vec{c}\right|=\sqrt{3^2+(-3)^2+2^2}=\sqrt{9+9+4}=\sqrt{22}$$

Hence, the unit vector along $2\vec{a}-\vec{b}+3\vec{c}$ is

$$\frac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|}=\frac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}}=\frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}.$$

Question 8:

Show that the points A (1, −2, −8), B (5, 0, −2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

Answer

The given points are A (1, −2, −8), B (5, 0, −2), and C (11, 3, 7).

$$\therefore \overrightarrow{AB}=(5-1)\hat{i}+(0+2)\hat{j}+(-2+8)\hat{k}=4\hat{i}+2\hat{j}+6\hat{k}$$

$$\overrightarrow{BC}=(11-5)\hat{i}+(3-0)\hat{j}+(7+2)\hat{k}=6\hat{i}+3\hat{j}+9\hat{k}$$

$$\overrightarrow{AC}=(11-1)\hat{i}+(3+2)\hat{j}+(7+8)\hat{k}=10\hat{i}+5\hat{j}+15\hat{k}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{4^2+2^2+6^2}=\sqrt{16+4+36}=\sqrt{56}=2\sqrt{14}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{6^2+3^2+9^2}=\sqrt{36+9+81}=\sqrt{126}=3\sqrt{14}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{10^2+5^2+15^2}=\sqrt{100+25+225}=\sqrt{350}=5\sqrt{14}$$

$$\therefore \left|\overrightarrow{AC}\right|=\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|$$

Thus, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda:1$. Then, we have:

$$\overrightarrow{OB}=\frac{\lambda\overrightarrow{OC}+\overrightarrow{OA}}{(\lambda+1)}$$

$$\Rightarrow 5\hat{i}-2\hat{k}=\frac{\lambda\left(11\hat{i}+3\hat{j}+7\hat{k}\right)+\left(\hat{i}-2\hat{j}-8\hat{k}\right)}{\lambda+1}$$

$$\Rightarrow(\lambda+1)\left(5\hat{i}-2\hat{k}\right)=11\lambda\hat{i}+3\lambda\hat{j}+7\lambda\hat{k}+\hat{i}-2\hat{j}-8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i}-2(\lambda+1)\hat{k}=(11\lambda+1)\hat{i}+(3\lambda-2)\hat{j}+(7\lambda-8)\hat{k}$$

On equating the corresponding components, we get:

$$5(\lambda+1)=11\lambda+1$$

$$\Rightarrow 5\lambda+5=11\lambda+1$$

$$\Rightarrow 6\lambda=4$$

$$\Rightarrow \lambda=\frac{4}{6}=\frac{2}{3}$$

Hence, point B divides AC in the ratio $2:3.$

Question 9:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\left(2\vec{a}+\vec{b}\right)\text{and}\left(\vec{a}-3\vec{b}\right)$ externally in the ratio 1: 2. Also, show that P is the mid point of the line segment RQ.

Answer

It is given that $\overrightarrow{OP}=2\vec{a}+\vec{b},\ \overrightarrow{OQ}=\vec{a}-3\vec{b}$.

It is given that point R divides a line segment joining two points P and Q externally in the ratio 1: 2. Then, on using the section formula, we get:

$$\overrightarrow{OR}=\frac{2\left(2\vec{a}+\vec{b}\right)-\left(\vec{a}-3\vec{b}\right)}{2-1}=\frac{4\vec{a}+2\vec{b}-\vec{a}+3\vec{b}}{1}=3\vec{a}+5\vec{b}$$

Therefore, the position vector of point R is $3\vec{a}+5\vec{b}$.

Position vector of the mid-point of RQ = $\dfrac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$=\frac{\left(\vec{a}-3\vec{b}\right)+\left(3\vec{a}+5\vec{b}\right)}{2}$$
$$=2\vec{a}+\vec{b}$$
$$=\overrightarrow{OP}$$

Hence, P is the mid-point of the line segment RQ.

Question 10:

The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$.
Find the unit vector parallel to its diagonal. Also, find its area.

Answer

Adjacent sides of a parallelogram are given as: $\vec{a}=2\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}-3\hat{k}$ Then, the diagonal of a parallelogram is given by $\vec{a}+\vec{b}$.

$$\vec{a}+\vec{b}=(2+1)\hat{i}+(-4-2)\hat{j}+(5-3)\hat{k}=3\hat{i}-6\hat{j}+2\hat{k}$$

Thus, the unit vector parallel to the diagonal is

$$\frac{\vec{a}+\vec{b}}{\left|\vec{a}+\vec{b}\right|}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{3^2+(-6)^2+2^2}}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4}}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{7}=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}.$$

$\therefore$ Area of parallelogram ABCD $=\left|\vec{a}\times\vec{b}\right|$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -4 & 5\\ 1 & -2 & -3\end{vmatrix}$$
$$=\hat{i}(12+10)-\hat{j}(-6-5)+\hat{k}(-4+4)$$
$$=22\hat{i}+11\hat{j}$$
$$=11\left(2\hat{i}+\hat{j}\right)$$
$$\therefore\left|\vec{a}\times\vec{b}\right|=11\sqrt{2^2+1^2}=11\sqrt{5}$$

Hence, the area of the parallelogram is $11\sqrt{5}$ square units.

Question 11:

Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Answer

Let a vector be equally inclined to axes OX, OY, and OZ at angle α.

Then, the direction cosines of the vector are cos α, cos α, and cos α.

Now,

$$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow 3\cos^2\alpha = 1$$

$$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$$

Hence, the direction cosines of the vector which are equally inclined to the axes are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Question 12:

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$,

and $\vec{c}.\vec{d} = 15$. Answer

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have:

$$\vec{d} \cdot \vec{a} = 0$$

$$\Rightarrow d_1 + 4d_2 + 2d_3 = 0 \quad \text{...(i)}$$

And,

$$\vec{d} \cdot \vec{b} = 0$$

$$\Rightarrow 3d_1 - 2d_2 + 7d_3 = 0 \quad \text{...(ii)}$$

Also, it is given that:

$$\vec{c} \cdot \vec{d} = 15$$

$$\Rightarrow 2d_1 - d_2 + 4d_3 = 15 \quad \text{...(iii)}$$

On solving (i), (ii), and (iii), we get:

$$d_1 = \frac{160}{3}, d_2 = -\frac{5}{3} \text{ and } d_3 = -\frac{70}{3}$$

$$\therefore \vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k} = \frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$$

Hence, the required vector is $\frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$.

Question 13:

The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ. Answer

$$\left(2\hat{i} + 4\hat{j} - 5\hat{k}\right) + \left(\lambda\hat{i} + 2\hat{j} + 3\hat{k}\right)$$
$$= (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

Therefore, unit vector along $\left(2\hat{i} + 4\hat{j} - 5\hat{k}\right) + \left(\lambda\hat{i} + 2\hat{j} + 3\hat{k}\right)$ is given as:

$$\frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 6^2 + (-2)^2}} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{4 + 4\lambda + \lambda^2 + 36 + 4}} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

Scalar product of $\left(\hat{i} + \hat{j} + \hat{k}\right)$ with this unit vector is 1.

$$\Rightarrow \left(\hat{i} + \hat{j} + \hat{k}\right) \cdot \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$
$$\Rightarrow \frac{(2+\lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$
$$\Rightarrow \sqrt{\lambda^2 + 4\lambda + 44} = \lambda + 6$$
$$\Rightarrow \lambda^2 + 4\lambda + 44 = (\lambda + 6)^2$$
$$\Rightarrow \lambda^2 + 4\lambda + 44 = \lambda^2 + 12\lambda + 36$$
$$\Rightarrow 8\lambda = 8$$
$$\Rightarrow \lambda = 1$$

Hence, the value of λ is 1.

Question 14:

If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$. Answer

Since $\vec{a},\vec{b},$ and $\vec{c}$ are mutually perpendicular vectors, we have

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0.$

It is given that:

$|\vec{a}|=|\vec{b}|=|\vec{c}|$

Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b},$ and $\vec{c}$ at angles $\theta_1,\ \theta_2,$ and θ_3 respectively. Then, we have:

$$\cos\theta_1=\frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$=\frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}\qquad\left[\vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{a}=0\right]$$

$$=\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_2=\frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|}$$

$$=\frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|}\qquad\left[\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{b}=0\right]$$

$$=\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_3=\frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}$$

$$=\frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}\qquad\left[\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}=0\right]$$

$$=\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Now, as $|\vec{a}| = |\vec{b}| = |\vec{c}|$, $\cos\theta_1 = \cos\theta_2 = \cos\theta_3$.

$\therefore \theta_1 = \theta_2 = \theta_3$

Hence, the vector $\left(\vec{a}+\vec{b}+\vec{c}\right)$ is equally inclined to $\vec{a}, \vec{b}$, and $\vec{c}$.

Question 15:

Prove that $\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right) = |\vec{a}|^2 + |\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Answer

$\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right) = |\vec{a}|^2 + |\vec{b}|^2$

$\Leftrightarrow \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2$ [Distributivity of scalar products over addition]

$\Leftrightarrow |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$ $\left[\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}\ \text{(Scalar product is commutative)}\right]$

$\Leftrightarrow 2\vec{a}\cdot\vec{b} = 0$

$\Leftrightarrow \vec{a}\cdot\vec{b} = 0$

$\therefore \vec{a}$ and $\vec{b}$ are perpendicular. $\left[\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}\ \text{(Given)}\right]$

Question 16:

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$ (B) $0 \leq \theta \leq \frac{\pi}{2}$

(C) $0 < \theta < \pi$ (D) $0 \leq \theta \leq \pi$

Answer

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

It is known that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

$\therefore \vec{a}\cdot\vec{b}\geq 0$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$

$\Rightarrow \cos\theta \geq 0 \qquad \left[|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}\right]$

$\Rightarrow 0\leq\theta\leq\frac{\pi}{2}$

Hence, $\vec{a}.\vec{b}\geq 0$ when $0\leq\theta\leq\frac{\pi}{2}$.

The correct answer is B.

Question 17:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors andθ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

(A) $\theta=\frac{\pi}{4}$ (B) $\theta=\frac{\pi}{3}$ (C) $\theta=\frac{\pi}{2}$ (D) $\theta=\frac{2\pi}{3}$

Answer

Let $\vec{a}$ and $\vec{b}$ be two unit vectors andθ be the angle between them.

Then, $|\vec{a}|=|\vec{b}|=1$.

Now, $\vec{a}+\vec{b}$ is a unit vector if $|\vec{a}+\vec{b}|=1$.

$\left|\vec{a}+\vec{b}\right|=1$

$\Rightarrow\left(\vec{a}+\vec{b}\right)^2=1$

$\Rightarrow\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)=1$

$\Rightarrow\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=1$

$\Rightarrow\left|\vec{a}\right|^2+2\vec{a}.\vec{b}+\left|\vec{b}\right|^2=1$

$\Rightarrow 1^2+2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta+1^2=1$

$\Rightarrow 1+2.1.1\cos\theta+1=1$

$\Rightarrow\cos\theta=-\frac{1}{2}$

$\Rightarrow\theta=\frac{2\pi}{3}$

Hence, $\vec{a}+\vec{b}$ is a unit vector if $\theta=\frac{2\pi}{3}$.

The correct answer is D.

Question 18:

The value of $\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{i}\times\hat{k}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$ is

(A) 0 (B) –1 (C) 1 (D) 3

Answer

$\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{i}\times\hat{k}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$

$=\hat{i}\cdot\hat{i}+\hat{j}\cdot\left(-\hat{j}\right)+\hat{k}\cdot\hat{k}$

$=1-\hat{j}\cdot\hat{j}+1$

$=1-1+1$

$=1$

The correct answer is C.

Question 19:

If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $\left|\vec{a}.\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$ when θ isequal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π Answer

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

$$|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$$
$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta$$
$$\Rightarrow \cos\theta = \sin\theta \qquad \left[|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}\right]$$
$$\Rightarrow \tan\theta = 1$$
$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, $|\vec{a}.\vec{b}| = |\vec{a}\times\vec{b}|$ when θ isequal to $\frac{\pi}{4}$

The correct answer is B.