Exercise 5.1

Question 1:

Prove that the function $f(x)=5x-3$ is continuous at $x=0, \text{at } x=-3 \text{ and at } x=5.$ Answer

$\text{The given function is } f(x)=5x-3$

$\text{At } x=0, f(0)=5\times 0-3=3$

$\lim_{x\to 0} f(x)=\lim_{x\to 0}(5x-3)=5\times 0-3=-3$

$\therefore \lim_{x\to 0} f(x)=f(0)$

Therefore, f is continuous at x = 0

$\text{At } x=-3, f(-3)=5\times(-3)-3=-18$

$\lim_{x\to -3} f(x)=\lim_{x\to -3}(5x-3)=5\times(-3)-3=-18$

$\therefore \lim_{x\to -3} f(x)=f(-3)$

Therefore, f is continuous at x = −3

$\text{At } x=5, f(x)=f(5)=5\times 5-3=25-3=22$

$\lim_{x\to 5} f(x)=\lim_{x\to 5}(5x-3)=5\times 5-3=22$

$\therefore \lim_{x\to 5} f(x)=f(5)$

Therefore, f is continuous at x = 5

Question 2:

Examine the continuity of the function $f(x)=2x^2-1 \text{ at } x=3$.

Answer

$\text{The given function is } f(x)=2x^2-1$

$\text{At } x=3, f(x)=f(3)=2\times 3^2-1=17$

$\lim_{x\to 3} f(x)=\lim_{x\to 3}(2x^2-1)=2\times 3^2-1=17$

$\therefore \lim_{x\to 3} f(x)=f(3)$

Thus, f is continuous at x = 3

Question 3:

Examine the following functions for continuity.

(a) $f(x)=x-5$ (b) $f(x)=\frac{1}{x-5}, x \neq 5$

(c) $f(x)=\frac{x^2-25}{x+5}, x \neq -5$ (d) $f(x)=|x-5|$ Answer

(a) The given function is $f(x)=x-5$

It is evident that f is defined at every real number k and its value at k is k − 5.

It is also observed that, $\lim_{x \to k} f(x)=\lim_{x \to k}(x-5)=k-5=f(k)$

$\therefore \lim_{x \to k} f(x)=f(k)$

Hence, f is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x)=\frac{1}{x-5}, x \neq 5$

For any real number k ≠ 5, we obtain

$\lim_{x \to k} f(x)=\lim_{x \to k} \frac{1}{x-5}=\frac{1}{k-5}$

$\text{Also}, f(k)=\frac{1}{k-5} \quad (\text{As } k \neq 5)$

$\therefore \lim_{x \to k} f(x)=f(k)$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(c) The given function is $f(x)=\frac{x^2-25}{x+5}, x \neq -5$

For any real number c ≠ −5, we obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{x^2-25}{x+5} = \lim_{x\to c} \frac{(x+5)(x-5)}{x+5} = \lim_{x\to c}(x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, \text{ if } x < 5 \\ x-5, \text{ if } x \geq 5 \end{cases}$

This function f is defined at all points of the real line.

Let c be a point on a real line. Then, c < 5 or c = 5 or c > 5

Case I: c < 5

Then, f (c) = 5 − c

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = 5-c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers less than 5.

Case II : c = 5

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5}(5-x) = (5-5) = 0$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}(x-5) = 0$$

$$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

Therefore, f is continuous at x = 5

Case III: c > 5

$$\text{Then, } f(c) = f(5) = c-5$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(x-5) = c-5$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x)=x^n$ is continuous at x = n, where n is a positive integer. Answer

The given function is f (x) = x^n

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

$\text{Then, } \lim_{x\to n} f(n)=\lim_{x\to n}(x^n)=n^n$

$\therefore \lim_{x\to n} f(x)=f(n)$

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by

$$f(x)=\begin{cases}x, & \text{if } x\le 1\\ 5, & \text{if } x>1\end{cases}$$

continuous at x = 0? At x = 1? At x = 2? Answer

The given function f is $f(x)=\begin{cases}x, & \text{if } x\le 1\\ 5, & \text{if } x>1\end{cases}$

At x = 0,

It is evident that f is defined at 0 and its value at 0 is 0.

$\text{Then, } \lim_{x\to 0} f(x)=\lim_{x\to 0} x=0$

$\therefore \lim_{x\to 0} f(x)=f(0)$

Therefore, f is continuous at x = 0

At x = 1,

f is defined at 1 and its value at 1 is 1. The left hand limit of f at x = 1 is,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x=1$

The right hand limit of f at x = 1 is,

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(5) = 5$

$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

Therefore, f is not continuous at x = 1

At x = 2,

f is defined at 2 and its value at 2 is 5.

$\text{Then, } \lim_{x\to 2} f(x) = \lim_{x\to 2}(5) = 5$

$\therefore \lim_{x\to 2} f(x) = f(2)$

Therefore, f is continuous at x = 2

Question 6:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

Answer

The given function f is $f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

It is evident that the given function f is defined at all the points of the real line. Let c be a point on the real line. Then, three cases arise.

(i) c < 2

(ii) c > 2

(iii) c = 2

Case (i) c < 2

$\text{Then, } f(c) = 2c+3$

$\lim_{x\to c} f(x) = \lim_{x\to c}(2x+3) = 2c+3$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x < 2

Case (ii) c > 2

$\text{Then, } f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(2x - 3) = 2c - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 2

Case (iii) c = 2

Then, the left hand limit of f at x = 2 is,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(2x + 3) = 2 \times 2 + 3 = 7$

The right hand limit of f at x = 2 is,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(2x - 3) = 2 \times 2 - 3 = 1$

It is observed that the left and right hand limit of f at x = 2 do not coincide. Therefore, f is not continuous at x = 2

Hence, x = 2 is the only point of discontinuity of f.

Question 7:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} |x| + 3, \text{if } x \le -3 \\ -2x, \text{if } -3 < x < 3 \\ 6x + 2, \text{if } x \ge 3 \end{cases}$$

Answer

The given function f is

$$f(x) = \begin{cases} |x| + 3 = -x + 3, \text{ if } x \le -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x + 2, \text{ if } x \ge 3 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$\text{If } c < -3, \text{ then } f(c) = -c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-x + 3) = -c + 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x < −3

Case II:

$$\text{If } c=-3,\ \text{then } f(-3)=-(-3)+3=6$$

$$\lim_{x\to -3^-} f(x)=\lim_{x\to -3^-}(-x+3)=-(-3)+3=6$$

$$\lim_{x\to -3^+} f(x)=\lim_{x\to -3^+}(-2x)=-2\times(-3)=6$$

$$\therefore \lim_{x\to -3} f(x)=f(-3)$$

Therefore, f is continuous at x = −3

Case III:

$$\text{If } -3<c<3,\ \text{then } f(c)=-2c \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(-2x)=-2c$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous in (−3, 3).

Case IV:

If c = 3, then the left hand limit of f at x = 3 is,

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(-2x)=-2\times 3=-6$$

The right hand limit of f at x = 3 is,

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(6x+2)=6\times 3+2=20$$

It is observed that the left and right hand limit of f at x = 3 do not coincide. Therefore, f is not continuous at x = 3

Case V:

$$\text{If } c>3,\ \text{then } f(c)=6c+2 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(6x+2)=6c+2$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x > 3

Hence, x = 3 is the only point of discontinuity of f.

Question 8:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}\frac{|x|}{x} \text{ if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$$

Answer

The given function f is $$f(x)=\begin{cases}\frac{|x|}{x} \text{ if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$$

It is known that, $x<0\Rightarrow|x|=-x \text{ and } x>0\Rightarrow|x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{|x|}{x}=\frac{-x}{x}=-1 \text{ if } x<0\\ 0, \text{ if } x=0\\ \frac{|x|}{x}=\frac{x}{x}=1, \text{ if } x>0\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$\text{If } c<0, \text{ then } f(c)=-1$

$\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x < 0

Case II:

If c = 0, then the left hand limit of f at x = 0 is,

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-1)=-1$

The right hand limit of f at x = 0 is,

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(1)=1$

It is observed that the left and right hand limit of f at x = 0 do not coincide. Therefore, f is not continuous at x = 0

Case III:

$\text{If } c > 0, \text{ then } f(c) = 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(1) = 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 0

Hence, x = 0 is the only point of discontinuity of f.

Question 9:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \frac{x}{|x|}, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

Answer

The given function f is $$f(x) = \begin{cases} \frac{x}{|x|}, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

$\Rightarrow f(x) = -1 \text{ for all } x \in \mathbf{R}$

Let c be any real number. Then, $\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}x+1, & \text{if } x\geq 1\\ x^2+1, & \text{if } x<1\end{cases}$$

Answer

The given function f is $f(x)=\begin{cases}x+1, & \text{if } x\geq 1\\ x^2+1, & \text{if } x<1\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<1, \text{ then } f(c)=c^2+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2+1\right)=c^2+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x < 1

Case II:

$$\text{If } c=1, \text{ then } f(c)=f(1)=1+1=2$$

The left hand limit of f at x = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^2+1\right)=1^2+1=2$$

The right hand limit of f at x = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x+1)=1+1=2$$

$$\therefore \lim_{x\to 1} f(x)=f(1)$$

Therefore, f is continuous at x = 1

Case III:

$$\text{If } c>1, \text{ then } f(c)=c+1$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x > 1

Hence, the given function f has no point of discontinuity.

Question 11:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$$

Answer

The given function f is $f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$\text{If } c<2, \text{ then } f(c)=c^3-3 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x^3-3)=c^3-3$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that x < 2

Case II:

$\text{If } c=2, \text{ then } f(c)=f(2)=2^3-3=5$

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x^3-3)=2^3-3=5$

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x^2+1)=2^2+1=5$

$\therefore \lim_{x\to 2} f(x)=f(2)$

Therefore, f is continuous at x = 2

Case III:

$\text{If } c>2, \text{ then } f(c)=c^2+1$

$\lim_{x\to c} f(x)=\lim_{x\to c}(x^2+1)=c^2+1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that x > 2

Thus, the given function f is continuous at every point on the real line. Hence, f has no point of discontinuity.

Question 12:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}x^{10}-1, & \text{if } x\le 1\\ x^2, & \text{if } x>1\end{cases}$$

Answer

The given function f is $f(x)=\begin{cases}x^{10}-1, & \text{if } x\le 1\\ x^2, & \text{if } x>1\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<1, \text{ then } f(c)=c^{10}-1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(x^{10}-1\right)=c^{10}-1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x < 1

Case II:

If c = 1, then the left hand limit of f at x = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$$

The right hand limit of f at x = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^2\right)=1^2=1$$

It is observed that the left and right hand limit of f at x = 1 do not coincide. Therefore, f is not continuous at x = 1

Case III:

$$\text{If } c>1, \text{ then } f(c)=c^2$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\right)=c^2$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity of f.

Question 13:

Is the function defined by

$$f(x)=\begin{cases}x+5, & \text{if } x\le 1\\ x-5, & \text{if } x>1\end{cases}$$

a continuous

function? Answer

The given function is $f(x)=\begin{cases}x+5, & \text{if } x\le 1\\ x-5, & \text{if } x>1\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<1,\ \text{then } f(c)=c+5 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+5)=c+5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x < 1

Case II:

$$\text{If } c=1,\ \text{then } f(1)=1+5=6$$

The left hand limit of f at x = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+5)=1+5=6$$

The right hand limit of f at x = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x-5)=1-5=-4$$

It is observed that the left and right hand limit of f at x = 1 do not

coincide. Therefore, f is not continuous at x = 1

Case III:

$$\text{If } c>1,\ \text{then } f(c)=c-5 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x-5)=c-5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observation, it can be concluded that x = 1 is the only point

of discontinuity of f.

Question 14:

Discuss the continuity of the function f, where f is defined by

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

Answer

The given function is

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

The given function is defined at all points of the interval [0, 10].

Let c be a point in the interval [0, 10].

Case I:

$$\text{If } 0\le c<1, \text{ then } f(c)=3 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(3)=3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous in the interval [0, 1).

Case II:

$$\text{If } c=1, \text{ then } f(3)=3$$

The left hand limit of f at x = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3)=3$$

The right hand limit of f at x = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4)=4$$

It is observed that the left and right hand limits of f at x = 1 do not coincide. Therefore, f is not continuous at x = 1

Case III:

$$\text{If } 1<c<3, \text{ then } f(c)=4 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(4)=4$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points of the interval (1, 3).

Case IV:

$$\text{If } c=3, \text{ then } f(c)=5$$

The left hand limit of f at x = 3 is,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(4) = 4$$

The right hand limit of f at x = 3 is,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(5) = 5$$

It is observed that the left and right hand limits of f at x = 3 do not coincide. Therefore, f is not continuous at x = 3

Case V:

$$\text{If } 3 < c \le 10, \text{ then } f(c) = 5 \text{ and } \lim_{x\to c} f(x) = \lim_{x\to c}(5) = 5$$

$$\lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at x = 1 and x = 3

Question 15:

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = 2c$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(2x) = 2c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x < 0

Case II:

$\text{If } c = 0, \text{ then } f(c) = f(0) = 0$

The left hand limit of f at x = 0 is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 2 \times 0 = 0$

The right hand limit of f at x = 0 is,

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at x = 0

Case III:

$\text{If } 0 < c < 1, \text{ then } f(x) = 0 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (0, 1).

Case IV:

$\text{If } c = 1, \text{ then } f(c) = f(1) = 0$

The left hand limit of f at x = 1 is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$

The right hand limit of f at x = 1 is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4$

It is observed that the left and right hand limits of f at x = 1 do not coincide. Therefore, f is not continuous at x = 1

Case V:

$\text{If } c < 1, \text{ then } f(c) = 4c \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Hence, f is not continuous only at x = 1

Question 16:

Discuss the continuity of the function f, where f is defined by

$$f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$$

Answer

The given function f is $$f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<-1, \text{ then } f(c)=-2 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(-2)=-2$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x < -1$

Case II:

$$\text{If } c=-1, \text{ then } f(c)=f(-1)=-2$$

The left hand limit of f at $x = -1$ is,

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}(-2)=-2$$

The right hand limit of f at $x = -1$ is,

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}(2x)=2\times(-1)=-2$$

$$\therefore \lim_{x\to -1} f(x)=f(-1)$$

Therefore, f is continuous at $x = -1$

Case III:

$$\text{If } -1<c<1, \text{ then } f(c)=2c$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x)=2c$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points of the interval $(-1, 1)$.

Case IV:

$\text{If } c = 1, \text{ then } f(c) = f(1) = 2 \times 1 = 2$

The left hand limit of f at x = 1 is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2 \times 1 = 2$

The right hand limit of f at x = 1 is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$

$\therefore \lim_{x \to 1} f(x) = f(c)$

Therefore, f is continuous at x = 2

Case V:

$\text{If } c > 1, \text{ then } f(c) = 2 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (2) = 2$

$\lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Question 17:

Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \leq 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

is continuous at x =

3. Answer

The given function f is $f(x) = \begin{cases} ax + 1, & \text{if } x \leq 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$

If f is continuous at x = 3, then

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$ ...(1)

Also,

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = 3a+1$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (bx+3) = 3b+3$

$f(3) = 3a+1$

Therefore, from (1), we obtain

$3a+1 = 3b+3 = 3a+1$

$\Rightarrow 3a+1 = 3b+3$

$\Rightarrow 3a = 3b+2$

$\Rightarrow a = b+\frac{2}{3}$

Therefore, the required relationship is given by, $a = b+\frac{2}{3}$

Question 18:

For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at x = 0? What about continuity at x = 1? Answer

The given function f is $f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$

If f is continuous at x = 0, then

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow \lim_{x\to 0^-} \lambda(x^2-2x) = \lim_{x\to 0^+} (4x+1) = \lambda(0^2-2\times 0)$

$\Rightarrow \lambda(0^2-2\times 0) = 4\times 0+1 = 0$

$\Rightarrow 0 = 1 = 0$, which is not possible

Therefore, there is no value of λ for which f is continuous at x = 0

At x = 1,

f (1) = 4x + 1 = 4 × 1 + 1 = 5

$\lim_{x\to 1}(4x+1) = 4\times 1+1 = 5$

$\therefore \lim_{x\to 1} f(x) = f(1)$

Therefore, for any values of λ, f is continuous at x = 1

Question 19:

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Answer

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$g(n) = n - [n] = n - n = 0$

The left hand limit of f at x = n is,

$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x - [x]) = \lim_{x\to n^-}(x) - \lim_{x\to n^-}[x] = n - (n-1) = 1$

The right hand limit of f at x = n is,

$\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x - [x]) = \lim_{x\to n^+}(x) - \lim_{x\to n^+}[x] = n - n = 0$

It is observed that the left and right hand limits of f at x = n do not coincide. Therefore, f is not continuous at x = n

Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at x = p? Answer

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at x = p

$$\text{At } x=\pi,\ f(x)=f(\pi)=\pi^2-\sin\pi+5=\pi^2-0+5=\pi^2+5$$

$$\text{Consider } \lim_{x\to\pi} f(x)=\lim_{x\to\pi}\left(x^2-\sin x+5\right)$$

$$\text{Put } x=\pi+h$$

$$\text{If } x\to\pi,\ \textit{then it is evident that } h\to 0$$

$$\therefore \lim_{x\to\pi} f(x)=\lim_{x\to\pi}\left(x^2-\sin x+5\right)$$

$$=\lim_{h\to 0}\left[(\pi+h)^2-\sin(\pi+h)+5\right]$$

$$=\lim_{h\to 0}(\pi+h)^2-\lim_{h\to 0}\sin(\pi+h)+\lim_{h\to 0}5$$

$$=(\pi+0)^2-\lim_{h\to 0}\left[\sin\pi\cosh+\cos\pi\sinh\right]+5$$

$$=\pi^2-\lim_{h\to 0}\sin\pi\cosh-\lim_{h\to 0}\cos\pi\sinh+5$$

$$=\pi^2-\sin\pi\cos 0-\cos\pi\sin 0+5$$

$$=\pi^2-0\times 1-(-1)\times 0+5$$

$$=\pi^2+5$$

$$\therefore \lim_{x\to\pi} f(x)=f(\pi)$$

Therefore, the given function f is continuous at x = ᴨ

Question 21:

Discuss the continuity of the following functions.

(a) f (x) = sin x + cos x

(b) f (x) = sin x − cos x

(c) f (x) = sin x × cos

x Answer

It is known that if g and h are two continuous functions, then

$g+h,\ g-h,\ \text{and } g.h$ are also continuous.

It has to proved first that g (x) = sin x and h (x) = cos x are continuous functions. Let g (x) = sin x

It is evident that g (x) = sin x is defined for every real number. Let c be a real number. Put x = c + h

If x → c, then h → 0

$g(c) = \sin c$

$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$

$= \lim_{h \to 0} \sin(c + h)$

$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$

$= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)$

$= \sin c \cos 0 + \cos c \sin 0$

$= \sin c + 0$

$= \sin c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is a continuous function.

Let h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x → c, then h →

0 h (c) = cos c

$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$

$= \lim_{h \to 0} \cos(c + h)$

$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$

$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$

$= \cos c \cos 0 - \sin c \sin 0$

$= \cos c \times 1 - \sin c \times 0$

$= \cos c$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) f (x) = g (x) + h (x) = sin x + cos x is a continuous function

(b) f (x) = g (x) − h (x) = sin x − cos x is a continuous function

(c) f (x) = g (x) × h (x) = sin x × cos x is a continuous function

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions, Answer

It is known that if g and h are two continuous functions, then

$(i)\ \frac{h(x)}{g(x)},\ g(x) \neq 0$ is continuous

$(ii)\ \frac{1}{g(x)},\ g(x) \neq 0$ is continuous

$(iii)\ \frac{1}{h(x)},\ h(x) \neq 0$ is continuous

It has to be proved first that g (x) = sin x and h (x) = cos x are continuous functions. Let g (x) = sin x

It is evident that g (x) = sin x is defined for every real number. Let c be a real number. Put x = c + h

If x $\rightarrow$c, then h $\rightarrow$0

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c} \sin x \\ &= \lim_{h\to 0} \sin(c+h) \\ &= \lim_{h\to 0} [\sin c\cos h + \cos c \sin h] \\ &= \lim_{h\to 0} (\sin c\cos h) + \lim_{h\to 0} (\cos c \sin h) \\ &= \sin c\cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$

$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, g is a continuous function.

Let h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x ® c, then h ®

0 h (c) = cos c

$$\lim_{x\to c} h(x) = \lim_{x\to c} \cos x$$
$$= \lim_{h\to 0} \cos(c+h)$$
$$= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$
$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, h (x) = cos x is continuous function.

It can be concluded that,

$$\text{cosec}\, x = \frac{1}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \text{cosec}\, x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cosecant is continuous except at x = np, n Î Z

$$\sec x = \frac{1}{\cos x},\ \cos x \neq 0 \text{ is continuous}$$
$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z}) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z})$

$$\cot x = \frac{\cos x}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \cot x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cotangent is continuous except at x = np, n Î Z

Question 23:

Find the points of discontinuity of f, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

Answer

The given function f is $f(x)=\begin{cases}\frac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\geq 0\end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c<0, \text{ then } f(c)=\frac{\sin c}{c} \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x < 0

Case II:

$$\text{If } c>0, \text{ then } f(c)=c+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x > 0

Case III:

$$\text{If } c=0, \text{ then } f(c)=f(0)=0+1=1$$

The left hand limit of f at x = 0 is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1$$

The right hand limit of f at x = 0 is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+1)=1$$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

Determine if f defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

is a continuous

function? Answer

The given function f is $$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c\neq 0, \text{ then } f(c)=c^2\sin\frac{1}{c}$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\sin\frac{1}{x}\right)=\left(\lim_{x\to c}x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right)=c^2\sin\frac{1}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x ≠ 0

Case II:

$$\text{If } c=0, \text{ then } f(0)=0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0$

$$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer

The given function f is $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$\text{If } c \neq 0, \text{ then } f(c) = \sin c - \cos c$

$\lim_{x \to c} f(x) = \lim_{x \to c} (\sin x - \cos x) = \sin c - \cos c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x ≠ 0

Case II:

$\text{If } c = 0, \text{ then } f(0) = -1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 26:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases} \qquad \text{at } x = \frac{\pi}{2}$$

Answer

The given function f is $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

The given function f is continuous at $x = \frac{\pi}{2}$, if f is defined at $x = \frac{\pi}{2}$ and if the value of the f at $x = \frac{\pi}{2}$ equals the limit of f at $x = \frac{\pi}{2}$.

It is evident that f is defined at $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=3$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

$$\text{Put } x=\frac{\pi}{2}+h$$

$$\text{Then, } x\to\frac{\pi}{2}\Rightarrow h\to 0$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{h\to 0}\frac{-\sin h}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of k is 6.

Question 27:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases} \qquad \text{at } x=2$$

Answer

The given function is $f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases}$

The given function f is continuous at x = 2, if f is defined at x = 2 and if the value of f at x = 2 equals the limit of f at x = 2

It is evident that f is defined at x = 2 and $f(2)=k(2)^2=4k$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$
$$\Rightarrow \lim_{x\to 2^-} (kx^2) = \lim_{x\to 2^+} (3) = 4k$$
$$\Rightarrow k\times 2^2 = 3 = 4k$$
$$\Rightarrow 4k = 3 = 4k$$
$$\Rightarrow 4k = 3$$
$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of k is $\frac{3}{4}$.

Question 28:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \qquad \text{at } x = \pi$$

Answer

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function f is continuous at x = p, if f is defined at x = p and if the value of f at x = p equals the limit of f at x = p

It is evident that f is defined at x = p and $f(\pi) = k\pi + 1$

$$\lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^+} f(x) = f(\pi)$$
$$\Rightarrow \lim_{x\to \pi^-} (kx+1) = \lim_{x\to \pi^+} \cos x = k\pi + 1$$
$$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1$$
$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$
$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of k is $-\frac{2}{\pi}$.

Question 29:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases}kx+1, & \text{if } x\le 5\\ 3x-5, & \text{if } x>5\end{cases} \qquad \text{at } x=5$$

Answer

The given function f is $f(x)=\begin{cases}kx+1, & \text{if } x\le 5\\ 3x-5, & \text{if } x>5\end{cases}$

The given function f is continuous at x = 5, if f is defined at x = 5 and if the value of f at x = 5 equals the limit of f at x = 5

It is evident that f is defined at x = 5 and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$
$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$
$$\Rightarrow 5k+1=15-5=5k+1$$
$$\Rightarrow 5k+1=10$$
$$\Rightarrow 5k=9$$
$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Question 30:

Find the values of a and b such that the function defined by

$$f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$$

is a continuous function.

Answer

The given function f is $f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$

It is evident that the given function f is defined at all points of the real line. If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at x = 2 and x = 10 Since f is continuous at x = 2, we obtain

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$

$\Rightarrow 5=2a+b=5$

$\Rightarrow 2a+b=5 \quad \ldots(1)$

Since f is continuous at x = 10, we obtain

$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$

$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$

$\Rightarrow 10a+b=21=21$

$\Rightarrow 10a+b=21 \quad \ldots(2)$

On subtracting equation (1) from equation (2), we obtain 8a = 16

⇒ a = 2

By putting a = 2 in equation (1), we obtain 2 × 2 + b = 5

⇒ 4 + b = 5

$\Rightarrow b = 1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by f (x) = cos (x^2) is a continuous function. Answer

The given function is f (x) = cos (x^2)

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where g (x) = cos x and h (x) = x^2

$$\left[\because (goh)(x) = g\left(h(x)\right) = g\left(x^2\right) = \cos\left(x^2\right) = f(x)\right]$$

It has to be first proved that g (x) = cos x and h (x) = x^2 are continuous functions. It is evident that g is defined for every real number.

Let c be a real number.

Then, g (c) = cos c

$$\text{Put } x = c + h$$

$$\text{If } x \to c, \text{ then } h \to 0$$

$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c} \cos x \\ &= \lim_{h\to 0} \cos(c+h) \\ &= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, g (x) = cos x is continuous function.

$h(x) = x^2$

Clearly, h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$$

$$\therefore \lim_{x \to k} h(x) = h(k)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function. Answer

The given function is $f(x) = |\cos x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where $g(x) = |x|$ and $h(x) = \cos x$

$$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)\right]$$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

$$\text{If } c < 0, \text{ then } g(c) = -c \text{ and } \lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that x < 0

Case II:

$$\text{If } c > 0, \text{ then } g(c) = c \text{ and } \lim_{x \to c} g(x) = \lim_{x \to c} x = c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that x > 0

Case III:

$$\text{If } c = 0, \text{ then } g(c) = g(0) = 0$$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$

$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points. h (x) = cos x

It is evident that h (x) = cos x is defined for every real number. Let c be a real number. Put x = c + h

If x → c, then h →

0 h (c) = cos c

$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h (x) = cos x is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\cos x)=|\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$ is a continuous function.

Answer

Let $f(x)=\sin|x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where $g(x)=|x|$ and $h(x)=\sin x$

$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$

It has to be proved first that $g(x)=|x|$ and $h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

$\text{If } c<0, \text{ then } g(c)=-c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that x < 0

Case II:

$\text{If } c>0, \text{ then } g(c)=c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c} x=c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that x > 0

Case III:

$\text{If } c=0, \text{ then } g(c)=g(0)=0$

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (-x) = 0$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x) = 0$

$\therefore \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} (x) = g(0)$

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points. h (x) = sin x

It is evident that h (x) = sin x is defined for every real number. Let c be a real number. Put x = c + k

If x → c, then k →

0 h (c) = sin c

$h(c) = \sin c$

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} \sin x \\ &= \lim_{k\to 0} \sin(c+k) \\ &= \lim_{k\to 0} [\sin c\cos k + \cos c \sin k] \\ &= \lim_{k\to 0} (\sin c\cos k) + \lim_{h\to 0} (\cos c \sin k) \\ &= \sin c\cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$

$\therefore \lim_{x\to c} h(x) = g(c)$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

Answer

The given function is $f(x) = |x| - |x+1|$

The two functions, g and h, are defined as

$g(x)=|x|$ and $h(x)=|x+1|$

Then, f = g − h

The continuity of g and h is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

$\text{If } c<0, \text{ then } g(c)=-c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that x < 0

Case II:

$\text{If } c>0, \text{ then } g(c)=c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c} x=c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that x > 0

Case III:

$\text{If } c=0, \text{ then } g(c)=g(0)=0$

$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$

$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$

$\therefore \lim_{x\to 0^-} g(x)=\lim_{x\to 0^+}(x)=g(0)$

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points.

$h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases}-(x+1), & \text{if, } x<-1\\ x+1, & \text{if } x\geq -1\end{cases}$$

Clearly, h is defined for every real number.

Let c be a real number.

Case I:

$$\text{If } c<-1, \text{ then } h(c)=-(c+1) \text{ and } \lim_{x\to c} h(x)=\lim_{x\to c}\left[-(x+1)\right]=-(c+1)$$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, h is continuous at all points x, such that x < −1

Case II:

$$\text{If } c>-1, \text{ then } h(c)=c+1 \text{ and } \lim_{x\to c} h(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, h is continuous at all points x, such that x > −1

Case III:

$$\text{If } c=-1, \text{ then } h(c)=h(-1)=-1+1=0$$

$$\lim_{x\to -1^-} h(x)=\lim_{x\to -1^-}\left[-(x+1)\right]=-(-1+1)=0$$

$$\lim_{x\to -1^+} h(x)=\lim_{x\to -1^+}(x+1)=(-1+1)=0$$

$$\therefore \lim_{x\to -1^-} h(x)=\lim_{h\to -1^+} h(x)=h(-1)$$

Therefore, h is continuous at x = −1

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, f = g − h is also a continuous function. Therefore, f has no point of discontinuity.

Exercise 5.2

Question 1:

Differentiate the functions with respect to x.

$\sin\left(x^2+5\right)$

Answer

$\text{Let } f(x)=\sin\left(x^2+5\right),\ u(x)=x^2+5, \text{ and } v(t)=\sin t$

$\text{Then, } (vou)(x)=v\left(u(x)\right)=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, f is a composite of two functions.

$\text{Put } t=u(x)=x^2+5$

Then, we obtain

$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$

$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$

$\text{Therefore, by chain rule, } \frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]&=\cos\left(x^2+5\right)\cdot\frac{d}{dx}\left(x^2+5\right)\\&=\cos\left(x^2+5\right)\cdot\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]\\&=\cos\left(x^2+5\right)\cdot[2x+0]\\&=2x\cos\left(x^2+5\right)\end{aligned}$$

Question 2:

Differentiate the functions with respect to x.

$\cos\left(\sin x\right)$

Answer

$\text{Let } f(x) = \cos(\sin x), u(x) = \sin x, \text{ and } v(t) = \cos t$

$\text{Then, } (vou)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

Thus, f is a composite function of two functions.

Put t = u (x) = sin x

$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$

$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$

By chain rule, $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$

Alternate method

$\frac{d}{dx}[\cos(\sin x)] = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$

Question 3:

Differentiate the functions with respect to x.

$\sin(ax+b)$

Answer

$\text{Let } f(x) = \sin(ax+b), \ u(x) = ax+b, \text{ and } v(t) = \sin t$

$\text{Then, } (vou)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = f(x)$

Thus, f is a composite function of two functions, u and v.

Put t = u (x) = ax + b

Therefore,

$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$

$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$

Hence, by chain rule, we obtain

$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a\cos(ax+b)$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right] = \cos(ax+b).\frac{d}{dx}(ax+b)$$
$$= \cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$= \cos(ax+b).(a+0)$$
$$= a\cos(ax+b)$$

Question 4:

Differentiate the functions with respect to x.

$$\sec\left(\tan\left(\sqrt{x}\right)\right)$$

Answer

$$\text{Let } f(x) = \sec\left(\tan\sqrt{x}\right), u(x) = \sqrt{x}, v(t) = \tan t, \text{and } w(s) = \sec s$$

$$\text{Then, } (wovou)(x) = w\left[v\left(u(x)\right)\right] = w\left[v\left(\sqrt{x}\right)\right] = w\left(\tan\sqrt{x}\right) = \sec\left(\tan\sqrt{x}\right) = f(x)$$

Thus, f is a composite function of three functions, u, v, and w.

$$\text{Put } s = v(t) = \tan t \text{ and } t = u(x) = \sqrt{x}$$

$$\text{Then, } \frac{dw}{ds} = \frac{d}{ds}(\sec s) = \sec s\tan s = \sec(\tan t).\tan(\tan t) \qquad [s = \tan t]$$
$$= \sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right) \qquad \left[t = \sqrt{x}\right]$$

$$\frac{ds}{dt} = \frac{d}{dt}(\tan t) = \sec^2 t = \sec^2\sqrt{x}$$

$$\frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}\cdot x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect to x.

$$\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}$$

Answer

The given function is $f\left(x\right)=\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}=\frac{g\left(x\right)}{h\left(x\right)}$, where g (x) = sin (ax + b) and h (x) = cos (cx + d)

$$\therefore f'=\frac{g'h-gh'}{h^2}$$

$$\text{Consider } g\left(x\right)=\sin\left(ax+b\right)$$

$$\text{Let } u\left(x\right)=ax+b, v\left(t\right)=\sin t$$

$$\text{Then, } \left(vou\right)\left(x\right)=v\left(u\left(x\right)\right)=v\left(ax+b\right)=\sin\left(ax+b\right)=g\left(x\right)$$

∴ g is a composite function of two functions, u and v.

$\text{Put } t = u(x) = ax + b$

$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$

$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$

Therefore, by chain rule, we obtain

$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a\cos(ax + b)$

$\text{Consider } h(x) = \cos(cx + d)$

$\text{Let } p(x) = cx + d,\ q(y) = \cos y$

$\text{Then, } (qop)(x) = q(p(x)) = q(cx + d) = \cos(cx + d) = h(x)$

∴h is a composite function of two functions, p and q.

Put y = p (x) = cx + d

$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx + d)$

$\frac{dy}{dx} = \frac{d}{dx}(cx + d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$

Therefore, by chain rule, we obtain

$h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx + d) \times c = -c\sin(cx + d)$

$$\therefore f' = \frac{a\cos(ax+b)\cdot\cos(cx+d) - \sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6:

Differentiate the functions with respect to x.

$\cos x^3.\sin^2(x^5)$

Answer

The given function is $\cos x^3.\sin^2(x^5)$.

$$\frac{d}{dx}\left[\cos x^3\cdot\sin^2(x^5)\right] = \sin^2(x^5)\times\frac{d}{dx}(\cos x^3) + \cos x^3\times\frac{d}{dx}\left[\sin^2(x^5)\right]$$

$$= \sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3) + \cos x^3\times 2\sin(x^5)\cdot\frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3\sin^2(x^5)\times 3x^2 + 2\sin x^5\cos x^3\cdot\cos x^5\times\frac{d}{dx}(x^5)$$

$$= -3x^2\sin x^3\cdot\sin^2(x^5) + 2\sin x^5\cos x^5\cos x^3\cdot\times 5x^4$$

$$= 10x^4\sin x^5\cos x^5\cos x^3 - 3x^2\sin x^3\sin^2(x^5)$$

Question 7:

Differentiate the functions with respect to x.

$2\sqrt{\cot(x^2)}$

Answer

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$

$$=2\cdot\frac{1}{2\sqrt{\cot\left(x^2\right)}}\times\frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$=\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times-\operatorname{cosec}^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$

$$=-\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{1}{\sin^2\left(x^2\right)}\times\left(2x\right)$$

$$=\frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\sin x^2}$$

$$=\frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\sin x^2}$$

$$=\frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 8:

Differentiate the functions with respect to x.

$$\cos\left(\sqrt{x}\right)$$

Answer

$$\text{Let } f\left(x\right)=\cos\left(\sqrt{x}\right)$$

$$\text{Also, let } u\left(x\right)=\sqrt{x}$$

$$\text{And, } v\left(t\right)=\cos t$$

$$\text{Then, } \left(vou\right)\left(x\right)=v\left(u\left(x\right)\right)$$

$$=v\left(\sqrt{x}\right)$$

$$=\cos\sqrt{x}$$

$$=f\left(x\right)$$

Clearly, f is a composite function of two functions, u and v, such that

$$t=u\left(x\right)=\sqrt{x}$$

$$\text{Then, } \frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$\text{And, } \frac{dv}{dt} = \frac{d}{dt}\left(\cos t\right) = -\sin t$$

$$= -\sin\left(\sqrt{x}\right)$$

By using chain rule, we obtain

$$\frac{dt}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}$$

$$= -\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$

$$= -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right] = -\sin\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= -\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= -\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9:

Prove that the function f given by

$f(x) = |x-1|,\ x \in \mathbf{R}$ is notdifferentiable at x = 1.

Answer

The given function is $f(x) = |x-1|,\ x \in \mathbf{R}$

It is known that a function f is differentiable at a point x = c in its domain if both $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal. To check the differentiability of the given function at x = 1, consider the left hand limit of f at x = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{|1+h-1|-|1-1|}{h}$$

$$=\lim_{h\to 0^-}\frac{|h|-0}{h}=\lim_{h\to 0^-}\frac{-h}{h}\qquad (h<0\Rightarrow |h|=-h)$$

$$=-1$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{|1+h-1|-|1-1|}{h}$$

$$=\lim_{h\to 0^+}\frac{|h|-0}{h}=\lim_{h\to 0^+}\frac{h}{h}\qquad (h>0\Rightarrow |h|=h)$$

$$=1$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

Question 10:

Prove that the greatest integer function defined by $f(x)=[x], 0<x<3$ is not differentiable at x = 1 and x = 2.

Answer

The given function f is $f(x)=[x], 0<x<3$

It is known that a function f is differentiable at a point x = c in its domain if both $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at x = 1, consider the left hand limit of f at x = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

To check the differentiability of the given function at x = 2, consider the left hand limit of f at x = 2

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^-}\frac{1-2}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^+}\frac{2-2}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 2 are not equal, f is not differentiable at x = 2

Exercise 5.3

Question 1:

Find $\frac{dy}{dx}$:

$2x+3y=\sin x$

Answer

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$

$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$

$$\therefore \frac{dy}{dx}=\frac{\cos x-2}{3}$$

Question 2:

Find $\frac{dy}{dx}$:

$2x+3y=\sin y$

Answer

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad [\text{By using chain rule}]$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Question 3:

Find $\frac{dy}{dx}$:

$ax+by^2=\cos y$

Answer

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad \ldots(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ $\ldots(2)$

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Question 4:

Find $\frac{dy}{dx}$:

$xy+y^2=\tan x+y$

Answer

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(xy + y^2\right) = \frac{d}{dx}\left(\tan x + y\right)$$

$$\Rightarrow \frac{d}{dx}\left(xy\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(\tan x\right) + \frac{dy}{dx}$$

$$\Rightarrow \left[y \cdot \frac{d}{dx}\left(x\right) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow y \cdot 1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \left(x + 2y - 1\right)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{\left(x + 2y - 1\right)}$$

Question 5:

Find $\frac{dy}{dx}$:

$x^2 + xy + y^2 = 100$

Answer

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(x^2 + xy + y^2\right) = \frac{d}{dx}\left(100\right)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(xy\right) + \frac{d}{dx}\left(y^2\right) = 0$$ [Derivative of constant function is 0]

$$\Rightarrow 2x+\left[y\cdot\frac{d}{dx}(x)+x\cdot\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0 \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow 2x+y\cdot 1+x\cdot\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y+(x+2y)\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

Question 6:

Find $\frac{dy}{dx}$:

$$x^3+x^2y+xy^2+y^3=81$$

Answer

The given relationship is $x^3+x^2y+xy^2+y^3=81$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(x^3+x^2y+xy^2+y^3\right)=\frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^2y\right)+\frac{d}{dx}\left(xy^2\right)+\frac{d}{dx}\left(y^3\right)=0$$

$$\Rightarrow 3x^2+\left[y\frac{d}{dx}\left(x^2\right)+x^2\frac{dy}{dx}\right]+\left[y^2\frac{d}{dx}(x)+x\frac{d}{dx}\left(y^2\right)\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow 3x^2+\left[y\cdot 2x+x^2\frac{dy}{dx}\right]+\left[y^2\cdot 1+x\cdot 2y\cdot\frac{dy}{dx}\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \left(x^2+2xy+3y^2\right)\frac{dy}{dx}+\left(3x^2+2xy+y^2\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{-\left(3x^2+2xy+y^2\right)}{\left(x^2+2xy+3y^2\right)}$$

Question 7:

Find $\frac{dy}{dx}$:

$$\sin^2 y+\cos xy=\pi$$

Answer

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}\left(\cos xy\right) = 0 \qquad \ldots(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}\left(\sin y\right) = 2\sin y \cos y \frac{dy}{dx} \qquad \ldots(2)$$

$$\frac{d}{dx}\left(\cos xy\right) = -\sin xy \frac{d}{dx}\left(xy\right) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$

$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy\frac{dy}{dx} \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx} - y\sin xy - x\sin xy\frac{dy}{dx} = 0$$

$$\Rightarrow \left(2\sin y\cos y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \left(\sin 2y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$

$$\therefore \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

Find $\frac{dy}{dx}$:

$\sin^2 x + \cos^2 y = 1$

Answer

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left(\cos^2 y\right)=0$$

$$\Rightarrow 2\sin x\cdot\frac{d}{dx}\left(\sin x\right)+2\cos y\cdot\frac{d}{dx}\left(\cos y\right)=0$$

$$\Rightarrow 2\sin x\cos x+2\cos y\left(-\sin y\right)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \sin 2x-\sin 2y\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}$:

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin y\right)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad \ldots(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{\left(1+x^2\right).\frac{d}{dx}(2x)-2x.\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right).2-2x.[0+2x]}{\left(1+x^2\right)^2}=\frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2} \quad ...(2)$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}$$

$$=\sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\frac{1-x^2}{1+x^2} \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

Question 10:

Find $\frac{dy}{dx}$:

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$$

Answer

The given relationship is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \tan y = \frac{3x - x^3}{1 - 3x^2} \quad \ldots(1)$$

$$\tan y = \frac{3\tan\frac{y}{3} - \tan^3\frac{y}{3}}{1 - 3\tan^2\frac{y}{3}} \quad \ldots(2)$$

It is known that,

Comparing equations (1) and (2), we obtain

$$x = \tan\frac{y}{3}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3}\cdot\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3}\cdot\frac{1}{3}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2\frac{y}{3}} = \frac{3}{1 + \tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1 + x^2}$$

Question 11:

Find $\frac{dy}{dx}$:

$$y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right), 0 < x < 1$$

x

Answer

The given relationship is,

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}} = \frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2} = x$$

Differentiating this relationship with respect to x, we obtain

$$\sec^2\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

Question 12:

Find $\frac{dy}{dx}$:

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0 < x < 1$$

Answer

The given relationship is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad \ldots(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{\left(1+x^2\right)^2-\left(1-x^2\right)^2}{\left(1+x^2\right)^2}} = \sqrt{\frac{4x^2}{\left(1+x^2\right)^2}} = \frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx} \qquad \ldots(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\left(1+x^2\right)\cdot\left(1-x^2\right)' - \left(1-x^2\right)\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2} \qquad [\text{Using quotient rule}]$$

$$= \frac{\left(1+x^2\right)(-2x) - \left(1-x^2\right)\cdot(2x)}{\left(1+x^2\right)^2}$$

$$= \frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$= \frac{-4x}{\left(1+x^2\right)^2} \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Alternate method

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\sin y=1-x^2$$

$$\Rightarrow \left(1+\sin y\right)x^2=1-\sin y$$

$$\Rightarrow x^2=\frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2=\frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x=\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x=\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\cdot\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1=\sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

$$\Rightarrow 1=\left[1+\tan^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]\cdot\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1=\left(1+x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 13:

Find $\frac{dy}{dx}$:

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right),-1<x<1$$

Answer

The given relationship is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\cdot\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y\cdot\frac{dy}{dx}=\frac{\left(1+x^2\right)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\frac{dy}{dx}=\frac{\left(1+x^2\right)\times 2-2x\cdot 2x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx}=-\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 14:

Find $\frac{dy}{dx}$:

$$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right),\ -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$$

Answer

lationship is $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y=2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\cos y\frac{dy}{dx}=2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow\sqrt{1-\sin^2 y}\frac{dy}{dx}=2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$

$$\Rightarrow\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx}=2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\frac{dy}{dx}$:

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right),0<x<\frac{1}{\sqrt{2}}$$

Answer

The given relationship is $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

Exercise 5.4

Question 1:

Differentiate the following w.r.t. x:

$\frac{e^x}{\sin x}$

Answer

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}\left(\sin x\right)}{\sin^2 x}$$
$$= \frac{\sin x.\left(e^x\right) - e^x \cdot \left(\cos x\right)}{\sin^2 x}$$
$$= \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$$

Question 2:

Differentiate the following w.r.t. x:

$e^{\sin^{-1} x}$

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1}x} \cdot \frac{d}{dx}\left(\sin^{-1}x\right)$$

$$= e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}, x \in (-1,1)$$

Question 2:

Show that the function given by f(x) = e^{2x} is strictly increasing on R. Answer

Let x_1 and x_2 be any two numbers in R.

Then, we have:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{2x_1} < e^{2x_2} \Rightarrow f(x_1) < f(x_2)$$

Hence, f is strictly increasing on R.

Question 3:

Differentiate the following w.r.t. x:

e^{x^3}

Answer

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right) = e^{x^3} \cdot \frac{d}{dx}\left(x^3\right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

Question 4:

Differentiate the following w.r.t. x:

$\sin\left(\tan^{-1}e^{-x}\right)$

Answer

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{1}{1+\left(e^{-x}\right)^2} \cdot \frac{d}{dx}\left(e^{-x}\right)$$

$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}\left(-x\right)$$

$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \times \left(-1\right)$$

$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following w.r.t. x:

$\log\left(\cos e^{x}\right)$

Answer

Let $y = \log\left(\cos e^{x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^{x}\right)\right]$$

$$= \frac{1}{\cos e^{x}} \cdot \frac{d}{dx}\left(\cos e^{x}\right)$$

$$= \frac{1}{\cos e^{x}} \cdot \left(-\sin e^{x}\right) \cdot \frac{d}{dx}\left(e^{x}\right)$$

$$= \frac{-\sin e^{x}}{\cos e^{x}} \cdot e^{x}$$

$$= -e^{x}\tan e^{x}, e^{x} \neq \left(2n+1\right)\frac{\pi}{2}, n \in \mathbf{N}$$

Question 6:

Differentiate the following w.r.t. x:

$e^x + e^{x^2} + ... + e^{x^5}$

Answer

$$\frac{d}{dx}\left(e^x + e^{x^2} + ... + e^{x^5}\right)$$
$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$
$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$
$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$
$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Question 7:

Differentiate the following w.r.t. x:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to x, we obtain

$$y^2 = e^{\sqrt{x}}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad [\text{By applying the chain rule}]$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}.\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}},\ x > 0$$

Question 8:

Differentiate the following w.r.t. x:

$\log(\log x), x > 1$

Answer

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$

$$= \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x}\cdot\frac{1}{x}$$

$$= \frac{1}{x\log x}, \text{ x > 1}$$

Question 9:

Differentiate the following w.r.t. x:

$\frac{\cos x}{\log x}, x > 0$

Answer

Let $y = \frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x) \times \log x - \cos x \times \frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{-\sin x \log x - \cos x \times \frac{1}{x}}{(\log x)^2}$$

$$= \frac{-[x \log x . \sin x + \cos x]}{x(\log x)^2}, x > 0$$

Question 10:

Differentiate the following w.r.t. x:

$\cos(\log x + e^x), x > 0$

Answer

Let $y = \cos(\log x + e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin(\log x + e^x) \cdot \frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)\right]$$

$$= -\sin(\log x + e^x) \cdot \left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right) \sin(\log x + e^x), x > 0$$

Exercise 5.5

Question 1:

Differentiate the function with respect to x.

$\cos x.\cos 2x.\cos 3x$

Answer

$\text{Let } y = \cos x.\cos 2x.\cos 3x$

Taking logarithm on both the sides, we obtain

$\log y = \log\left(\cos x.\cos 2x.\cos 3x\right)$

$\Rightarrow \log y = \log\left(\cos x\right) + \log\left(\cos 2x\right) + \log\left(\cos 3x\right)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}\cdot\frac{d}{dx}\left(\cos x\right) + \frac{1}{\cos 2x}\cdot\frac{d}{dx}\left(\cos 2x\right) + \frac{1}{\cos 3x}\cdot\frac{d}{dx}\left(\cos 3x\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x}\cdot\frac{d}{dx}\left(2x\right) - \frac{\sin 3x}{\cos 3x}\cdot\frac{d}{dx}\left(3x\right)\right]$$

$$\therefore \frac{dy}{dx} = -\cos x.\cos 2x.\cos 3x\left[\tan x + 2\tan 2x + 3\tan 3x\right]$$

Question 2:

Differentiate the function with respect to x.

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Answer

$$\text{Let } y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y = \frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\right]$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}\cdot\frac{d}{dx}(x-1) + \frac{1}{x-2}\cdot\frac{d}{dx}(x-2) - \frac{1}{x-3}\cdot\frac{d}{dx}(x-3) \\ -\frac{1}{x-4}\cdot\frac{d}{dx}(x-4) - \frac{1}{x-5}\cdot\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Question 3:

Differentiate the function with respect to x.

$(\log x)^{\cos x}$

Answer

Let $y = (\log x)^{\cos x}$

Taking logarithm on both the sides, we obtain

$\log y = \cos x \cdot \log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{1}{y}\cdot\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

Question 4:

Differentiate the function with respect to x.

$x^x-2^{\sin x}$

Answer

Let $y=x^x-2^{\sin x}$

Also, let $x^x=u$ and $2^{\sin x}=v$

$\therefore y=u-v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$$

$u = x^x$

Taking logarithm on both the sides, we obtain

$\log u=x\log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\left[\frac{d}{dx}(x)\times\log x+x\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log x+x\times\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^x(\log x+1)$$

$$\Rightarrow\frac{du}{dx}=x^x(1+\log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to x, we obtain

$\log v = \sin x \cdot \log 2$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$$

Question 5:

Differentiate the function with respect to x.

$(x+3)^2 .(x+4)^3 .(x+5)^4$

Answer

Let $y = (x+3)^2 .(x+4)^3 .(x+5)^4$

Taking logarithm on both the sides, we obtain

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$

$$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2\cdot\frac{1}{x+3}\cdot\frac{d}{dx}(x+3) + 3\cdot\frac{1}{x+4}\cdot\frac{d}{dx}(x+4) + 4\cdot\frac{1}{x+5}\cdot\frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 \cdot \left[2(x^2+9x+20) + 3(x^2+8x+15) + 4(x^2+7x+12)\right]$$

$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 (9x^2+70x+133)$$

Question 6:

Differentiate the function with respect to x.

$$\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

Answer

$$\text{Let } y=\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

$$\text{Also, let } u=\left(x+\frac{1}{x}\right)^{x} \text{ and } v=x^{\left(1+\frac{1}{x}\right)}$$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(1)$$

$$\text{Then, } u=\left(x+\frac{1}{x}\right)^{x}$$

$$\Rightarrow \log u=\log\left(x+\frac{1}{x}\right)^{x}$$

$$\Rightarrow \log u=x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow\frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] \qquad ...(2)$$

$$v=x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow\log v=\log\left[x^{\left(1+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\log v=\left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(1+\frac{1}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)\qquad\ldots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

Question 7:

Differentiate the function with respect to x.

$$(\log x)^x+x^{\log x}$$

Answer

$$\text{Let } y=(\log x)^x+x^{\log x}$$

$$\text{Also, let } u=(\log x)^x \text{ and } v=x^{\log x}$$

$$\therefore y=u+v$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad\ldots(1)$$

$u = (\log x)^x$

$$\Rightarrow\log u=\log\left[(\log x)^x\right]$$

$$\Rightarrow\log u=x\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x\cdot\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log(\log x)+x\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\frac{\log(\log x)\cdot\log x+1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]\qquad\ldots(2)$$

$$v=x^{\log x}$$

$$\Rightarrow\log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow\log v=\log x\log x=(\log x)^{2}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^{2}\right]$$

$$\Rightarrow\frac{1}{v}\cdot\frac{dv}{dx}=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dv}{dx}=2v(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x-1}\cdot\log x\qquad\ldots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]+2x^{\log x-1}\cdot\log x$$

Question 8:

Differentiate the function with respect to x.

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Answer

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

$u = (\sin x)^x$

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\times\log(\sin x) + x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1\cdot\log(\sin x) + x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x(x\cot x + \log\sin x) \qquad \ldots(2)$$

$v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \qquad \ldots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\sin x)^x(x\cot x + \log\sin x) + \frac{1}{2\sqrt{x-x^2}}$$

Question 9:

Differentiate the function with respect to x.

$x^{\sin x}+(\sin x)^{\cos x}$

Answer

$\text{Let } y=x^{\sin x}+(\sin x)^{\cos x}$

$\text{Also, let } u=x^{\sin x} \text{ and } v=(\sin x)^{\cos x}$

$\therefore y=u+v$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \ldots(1)$

$u=x^{\sin x}$

$\Rightarrow \log u=\log\left(x^{\sin x}\right)$

$\Rightarrow \log u=\sin x \log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x)\cdot\log x+\sin x\cdot\frac{d}{dx}(\log x)$

$\Rightarrow \frac{du}{dx}=u\left[\cos x\log x+\sin x\cdot\frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx}=x^{\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right] \quad \ldots(2)$

$v=(\sin x)^{\cos x}$

$\Rightarrow \log v=\log(\sin x)^{\cos x}$

$\Rightarrow \log v=\cos x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}(\cos x)\times\log(\sin x)+\cos x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow\frac{dv}{dx}=v\left[-\sin x.\log(\sin x)+\cos x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\cot x\cos x\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[\cot x\cos x-\sin x\log\sin x\right] \quad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}\left[\cos x\cot x-\sin x\log\sin x\right]$$

Question 10:

Differentiate the function with respect to x.

$$x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

Answer

$$\text{Let } y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

$$\text{Also, let } u=x^{x\cos x} \text{ and } v=\frac{x^2+1}{x^2-1}$$

$$\therefore y=u+v$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \ldots(1)$$

$$u=x^{x\cos x}$$

$$\Rightarrow\log u=\log\left(x^{x\cos x}\right)$$

$$\Rightarrow\log u=x\cos x\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\cdot\cos x\cdot\log x+x\cdot\frac{d}{dx}(\cos x)\cdot\log x+x\cos x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1\cdot\cos x\cdot\log x+x\cdot(-\sin x)\log x+x\cos x\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left(\cos x\log x-x\sin x\log x+\cos x\right)$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right] \quad \ldots(2)$$

$$v=\frac{x^2+1}{x^2-1}$$

$$\Rightarrow\log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{-4x}{\left(x^2-1\right)^2} \quad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]-\frac{4x}{\left(x^2-1\right)^2}$$

Question 11:

Differentiate the function with respect to x.

$$(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$$

Answer

$\text{Let } y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

$\text{Also, let } u = (x\cos x)^x \text{ and } v = (x\sin x)^{\frac{1}{x}}$

$\therefore y = u + v$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots(1)$

$u = (x\cos x)^x$

$\Rightarrow \log u = \log(x\cos x)^x$

$\Rightarrow \log u = x\log(x\cos x)$

$\Rightarrow \log u = x[\log x + \log\cos x]$

$\Rightarrow \log u = x\log x + x\log\cos x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x\log x) + \frac{d}{dx}(x\log\cos x)$

$\Rightarrow \frac{du}{dx} = u\left[\left\{\log x\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log x)\right\} + \left\{\log\cos x\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log\cos x)\right\}\right]$

$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[\left(\log x\cdot 1 + x\cdot\frac{1}{x}\right) + \left\{\log\cos x\cdot 1 + x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)\right\}\right]$

$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[(\log x + 1) + \left\{\log\cos x + \frac{x}{\cos x}\cdot(-\sin x)\right\}\right]$

$\Rightarrow \frac{du}{dx} = (x\cos x)^x[(1+\log x) + (\log\cos x - x\tan x)]$

$\Rightarrow \frac{du}{dx} = (x\cos x)^x[1 - x\tan x + (\log x + \log\cos x)]$

$\Rightarrow \frac{du}{dx} = (x\cos x)^x[1 - x\tan x + \log(x\cos x)] \quad \ldots(2)$

$$v = (x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log (x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x}\log (x\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}(\log x + \log \sin x)$$

$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log \sin x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\frac{d}{dx}(\log x)\right] + \left[\log(\sin x)\cdot\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{x}\cdot\frac{1}{x}\right] + \left[\log(\sin x)\cdot\left(-\frac{1}{x^2}\right) + \frac{1}{x}\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1-\log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x - \log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x) + x\cot x}{x^2}\right] \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

Question 12:

Find $\frac{dy}{dx}$ of function.

$x^y + y^x = 1$

Answer

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes u + v = 1

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \ldots(1)$$

$$u = x^y$$

$$\Rightarrow \log u = \log\left(x^y\right)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad \ldots(2)$$

$$v = y^x$$

$$\Rightarrow \log v = \log\left(y^x\right)$$

$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow\left(x^y\log x+xy^{x-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Question 13:

Find $\frac{dy}{dx}$ of function.

$y^x=x^y$

Answer

The given function is $y^x=x^y$

Taking logarithm on both the sides, we obtain

$x\log y=y\log x$

Differentiating both sides with respect to x, we obtain

$$\log y\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log y)=\log x\cdot\frac{d}{dx}(y)+y\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\log y\cdot 1+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\log x\cdot\frac{dy}{dx}+y\cdot\frac{1}{x}$$

$$\Rightarrow\log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

Question 14:

$(\cos x)^y=(\cos y)^x$

Answer

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$y \log \cos x = x \log \cos y$

Differentiating both sides, we obtain

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) = \log \cos y + \frac{x}{\cos y}(-\sin y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$

$$\Rightarrow (\log \cos x + x \tan y) \frac{dy}{dx} = y \tan x + \log \cos y$$

$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

Question 15:

Find $\frac{dy}{dx}$ of function.

$xy = e^{(x-y)}$

Answer

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy) = \log\left(e^{x-y}\right)$$

$$\Rightarrow \log x + \log y = (x-y) \log e$$

$$\Rightarrow \log x + \log y = (x-y) \times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$

. Answer

The given relationship is $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}\left[f(x)\right]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log\left(1+x^2\right)+\frac{d}{dx}\log\left(1+x^4\right)+\frac{d}{dx}\log\left(1+x^8\right)$$

$$\Rightarrow\frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}\left(1+x^2\right)+\frac{1}{1+x^4}\cdot\frac{d}{dx}\left(1+x^4\right)+\frac{1}{1+x^8}\cdot\frac{d}{dx}\left(1+x^8\right)$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}\cdot4x^3+\frac{1}{1+x^8}\cdot8x^7\right]$$

$$\therefore f'(x)=(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

$$\text{Hence, } f'(1)=(1+1)\left(1+1^2\right)\left(1+1^4\right)\left(1+1^8\right)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+1^4}+\frac{8\times1^7}{1+1^8}\right]$$

$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

Question 17:

Differentiate $\left(x^5-5x+8\right)\left(x^3+7x+9\right)$ in three ways mentioned below

(i) By using product rule.

(ii) By expanding the product to obtain a single polynomial. (iii By logarithmic differentiation.

Do they all give the same answer? Answer

$$\text{Let } y=\left(x^5-5x+8\right)\left(x^3+7x+9\right)$$

(i)

Let $x^2-5x+8=u$ and $x^3+7x+9=v$

$\therefore y=uv$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot v+u\cdot\frac{dv}{dx}$ (By using product rule)

$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^2-5x+8\right)\cdot\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\cdot\frac{d}{dx}\left(x^3+7x+9\right)$

$\Rightarrow \frac{dy}{dx}=(2x-5)\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\left(3x^2+7\right)$

$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+x^2\left(3x^2+7\right)-5x\left(3x^2+7\right)+8\left(3x^2+7\right)$

$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4+7x^2\right)-15x^3-35x+24x^2+56$

$\therefore \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$

(ii)

$y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

$=x^2\left(x^3+7x+9\right)-5x\left(x^3+7x+9\right)+8\left(x^3+7x+9\right)$

$=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$

$=x^5-5x^4+15x^3-26x^2+11x+72$

$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(x^5-5x^4+15x^3-26x^2+11x+72\right)$

$=\frac{d}{dx}\left(x^5\right)-5\frac{d}{dx}\left(x^4\right)+15\frac{d}{dx}\left(x^3\right)-26\frac{d}{dx}\left(x^2\right)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)$

$=5x^4-5\times4x^3+15\times3x^2-26\times2x+11\times1+0$

$=5x^4-20x^3+45x^2-52x+11$

(iii) $y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

Taking logarithm on both the sides, we obtain

$\log y=\log\left(x^2-5x+8\right)+\log\left(x^3+7x+9\right)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\cdot\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times\left(2x-5\right)+\frac{1}{x^3+7x+9}\times\left(3x^2+7\right)\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{\left(2x-5\right)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)\left(x^3+7x+9\right)}\right]$$

$$\Rightarrow\frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$

$$\Rightarrow\frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$

$$\Rightarrow\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the results of

$\frac{dy}{dx}$ are same.

Question 18:

If u, v and w are functions of x, then show that

$$\frac{d}{dx}\left(u.v.w\right)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer

Let $y=u.v.w=u.\left(v.w\right)$

By applying product rule, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}.(v.w)+u.\frac{d}{dx}(v\cdot w)$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v\cdot w+u\left[\frac{dv}{dx}\cdot w+v\cdot\frac{dw}{dx}\right] \qquad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y=u.v.w$, we obtain

$$\log y=\log u+\log v+\log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\log u)+\frac{d}{dx}(\log v)+\frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx}=y\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=u.v.w.\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

Exercise 5.6

Question 1:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 2at^2,\ y = at^4$

Answer

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a \cdot \frac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a \cdot \frac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

x = a cos θ, y = b cos θ Answer

The given equations are x = a cos θ and y = b cos θ

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right) = a\left(-\sin\theta\right) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos\theta\right) = b\left(-\sin\theta\right) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

x = sin t, y = cos 2t

Answer

The given equations are x = sin t and y = cos 2t

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t \cos t}{\cos t} = -4\sin t$$

Question 4:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 4t,\ y = \frac{4}{t}$

Answer

The given equations are $x = 4t$ and $y = \frac{4}{t}$

$$\frac{dx}{dt}=\frac{d}{dt}(4t)=4$$
$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{4}{t}\right)=4\cdot\frac{d}{dt}\left(\frac{1}{t}\right)=4\cdot\left(\frac{-1}{t^2}\right)=\frac{-4}{t^2}$$
$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-4}{t^2}\right)}{4}=\frac{-1}{t^2}$$

Question 5:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x=\cos\theta-\cos 2\theta,\ y=\sin\theta-\sin 2\theta$

Answer

The given equations are $x=\cos\theta-\cos 2\theta$ and $y=\sin\theta-\sin 2\theta$

$$\text{Then, } \frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos 2\theta)=\frac{d}{d\theta}(\cos\theta)-\frac{d}{d\theta}(\cos 2\theta)$$
$$=-\sin\theta-(-2\sin 2\theta)=2\sin 2\theta-\sin\theta$$
$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin 2\theta)=\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\sin 2\theta)$$
$$=\cos\theta-2\cos 2\theta$$
$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$$

Question 6:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$

Answer

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0 + (-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1 - \cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Answer

The given equations are $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}\left(\sin^3 t\right) - \sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cdot\frac{d}{dt}\left(\sin t\right) - \sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}\cdot\sin^2 t\cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}}.\left(-2\sin 2t\right)}{\cos 2t}$$

$$= \frac{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right) - \cos^3 t\cdot\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.3\cos^2 t\cdot\frac{d}{dt}\left(\cos t\right) - \cos^3 t.\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}.\cos^2 t\left(-\sin t\right) - \cos^3 t\cdot\frac{1}{2\sqrt{\cos 2t}}.\left(-2\sin 2t\right)}{\cos 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t\sin 2t}{\cos 2t\cdot\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t\left(2\sin t \cos t\right)}{3\cos 2t \sin^2 t \cos t + \sin^3 t\left(2\sin t \cos t\right)}$$

$$= \frac{\sin t \cos t\left[-3\cos 2t.\cos t + 2\cos^3 t\right]}{\sin t \cos t\left[3\cos 2t \sin t + 2\sin^3 t\right]}$$

$$= \frac{\left[-3\left(2\cos^2 t - 1\right)\cos t + 2\cos^3 t\right]}{\left[3\left(1 - 2\sin^2 t\right)\sin t + 2\sin^3 t\right]} \qquad \left[\begin{matrix}\cos 2t = \left(2\cos^2 t - 1\right), \\ \cos 2t = \left(1 - 2\sin^2 t\right)\end{matrix}\right]$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-\cos 3t}{\sin 3t} \qquad \left[\begin{matrix}\cos 3t = 4\cos^3 t - 3\cos t, \\ \sin 3t = 3\sin t - 4\sin^3 t\end{matrix}\right]$$

$$= -\cot 3t$$

Question 8:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = a\left(\cos t + \log \tan \frac{t}{2}\right),\ y = a\sin t$$

Answer

The given equations are $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $y = a\sin t$

$$\text{Then, } \frac{dx}{dt} = a\cdot\left[\frac{d}{dt}(\cos t)+\frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}\cdot\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t+\cot\frac{t}{2}\cdot\sec^2\frac{t}{2}\cdot\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t+\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\times\frac{1}{\cos^2\frac{t}{2}}\times\frac{1}{2}\right]$$

$$= a\left[-\sin t+\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left(-\sin t+\frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t+1}{\sin t}\right)$$

$$= a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\sec\theta,\ y = b\tan\theta$

Answer

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

$$\text{Then, } \frac{dx}{d\theta} = a\cdot\frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b\cdot\frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a}\times\frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Answer

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta\cdot\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Question 11:

If $x=\sqrt{a^{\sin^{-1}t}}$, $y=\sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx}=-\frac{y}{x}$

Answer

The given equations are $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$\Rightarrow x=\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}$ and $y=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$

$\Rightarrow x=a^{\frac{1}{2}\sin^{-1}t}$ and $y=a^{\frac{1}{2}\cos^{-1}t}$

Consider $x=a^{\frac{1}{2}\sin^{-1}t}$

Taking logarithm on both the sides, we obtain

$\log x=\frac{1}{2}\sin^{-1}t\log a$

$\therefore \frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\sin^{-1}t\right)$

$\Rightarrow \frac{dx}{dt}=\frac{x}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$

$\Rightarrow \frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$

Then, consider $y=a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both the sides, we obtain

$\log y=\frac{1}{2}\cos^{-1}t\log a$

$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\cos^{-1}t\right)$

$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$

$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}.$$

Hence, proved.

Exercise 5.7

Question 1:

Find the second order derivatives of the function.

$x^2 + 3x + 2$

Answer

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Question 2:

Find the second order derivatives of the function.

x^{20}

Answer

Let $y = x^{20}$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{20}\right) = 20x^{19}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(20x^{19}\right) = 20\frac{d}{dx}\left(x^{19}\right) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

Question 3:

Find the second order derivatives of the function.

$x \cdot \cos x$

Answer

Let $y = x \cdot \cos x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x\cdot\cos x)=\cos x\cdot\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x\cdot 1+x(-\sin x)=\cos x-x\sin x$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}[\cos x-x\sin x]=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)$$

$$=-\sin x-\left[\sin x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\sin x)\right]$$

$$=-\sin x-(\sin x+x\cos x)$$

$$=-(x\cos x+2\sin x)$$

Question 4:

Find the second order derivatives of the function.

$\log x$

Answer

Let $y=\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)=\frac{1}{x}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function.

$x^3\log x$

Answer

Let $y=x^3\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^3\log x\right]=\log x\cdot\frac{d}{dx}\left(x^3\right)+x^3\cdot\frac{d}{dx}\left(\log x\right)$$
$$=\log x\cdot 3x^2+x^3\cdot\frac{1}{x}=\log x\cdot 3x^2+x^2$$
$$=x^2\left(1+3\log x\right)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[x^2\left(1+3\log x\right)\right]$$
$$=\left(1+3\log x\right)\cdot\frac{d}{dx}\left(x^2\right)+x^2\frac{d}{dx}\left(1+3\log x\right)$$
$$=\left(1+3\log x\right)\cdot 2x+x^2\cdot\frac{3}{x}$$
$$=2x+6x\log x+3x$$
$$=5x+6x\log x$$
$$=x\left(5+6\log x\right)$$

Question 6:

Find the second order derivatives of the function.

$e^x\sin 5x$

Answer

Let $y=e^x\sin 5x$

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x\sin 5x\right)=\sin 5x\cdot\frac{d}{dx}\left(e^x\right)+e^x\frac{d}{dx}\left(\sin 5x\right)$$
$$=\sin 5x\cdot e^x+e^x\cdot\cos 5x\cdot\frac{d}{dx}\left(5x\right)=e^x\sin 5x+e^x\cos 5x\cdot 5$$
$$=e^x\left(\sin 5x+5\cos 5x\right)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x\left(\sin 5x+5\cos 5x\right)\right]$$
$$=\left(\sin 5x+5\cos 5x\right)\cdot\frac{d}{dx}\left(e^x\right)+e^x\cdot\frac{d}{dx}\left(\sin 5x+5\cos 5x\right)$$
$$=\left(\sin 5x+5\cos 5x\right)e^x+e^x\left[\cos 5x\cdot\frac{d}{dx}\left(5x\right)+5\left(-\sin 5x\right)\cdot\frac{d}{dx}\left(5x\right)\right]$$
$$=e^x\left(\sin 5x+5\cos 5x\right)+e^x\left(5\cos 5x-25\sin 5x\right)$$
$$=e^x\left(10\cos 5x-24\sin 5x\right)=2e^x\left(5\cos 5x-12\sin 5x\right)$$

Then,

Question 7:

Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Answer

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cdot\cos 3x\right) = \cos 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}\left(\cos 3x\right)$$

$$= \cos 3x\cdot e^{6x}\cdot\frac{d}{dx}\left(6x\right) + e^{6x}\cdot\left(-\sin 3x\right)\cdot\frac{d}{dx}\left(3x\right)$$

$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \quad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6\cdot\frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3\cdot\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$= 6\cdot\left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3\cdot\left[\sin 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}\left(\sin 3x\right)\right] \quad [\text{Using } (1)]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x\cdot e^{6x}\cdot 6 + e^{6x}\cdot\cos 3x\cdot 3\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Question 8:

Find the second order derivatives of the function.

$\tan^{-1} x$

Answer

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}\left(1+x^2\right)^{-1}=(-1)\cdot\left(1+x^2\right)^{-2}\cdot\frac{d}{dx}\left(1+x^2\right)$$

$$=\frac{-1}{\left(1+x^2\right)^2}\times 2x=\frac{-2x}{\left(1+x^2\right)^2}$$

Question 9:

Find the second order derivatives of the function.

$\log\left(\log x\right)$

Answer

Let $y=\log\left(\log x\right)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\log x\right)\right]=\frac{1}{\log x}\cdot\frac{d}{dx}\left(\log x\right)=\frac{1}{x\log x}=\left(x\log x\right)^{-1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\left(x\log x\right)^{-1}\right]=(-1)\cdot\left(x\log x\right)^{-2}\cdot\frac{d}{dx}\left(x\log x\right)$$

$$=\frac{-1}{\left(x\log x\right)^2}\cdot\left[\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}\left(\log x\right)\right]$$

$$=\frac{-1}{\left(x\log x\right)^2}\cdot\left[\log x\cdot 1+x\cdot\frac{1}{x}\right]=\frac{-\left(1+\log x\right)}{\left(x\log x\right)^2}$$

Question 10:

Find the second order derivatives of the function.

$\sin\left(\log x\right)$

Answer

Let $y=\sin\left(\log x\right)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\left(\log x\right)\right]=\cos\left(\log x\right)\cdot\frac{d}{dx}\left(\log x\right)=\frac{\cos\left(\log x\right)}{x}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos\left(\log x\right)}{x}\right]$$

$$=\frac{x\cdot\frac{d}{dx}\left[\cos\left(\log x\right)\right]-\cos\left(\log x\right)\cdot\frac{d}{dx}\left(x\right)}{x^2}$$

$$=\frac{x\cdot\left[-\sin\left(\log x\right)\cdot\frac{d}{dx}\left(\log x\right)\right]-\cos\left(\log x\right).1}{x^2}$$

$$=\frac{-x\sin\left(\log x\right).\frac{1}{x}-\cos\left(\log x\right)}{x^2}$$

$$=\frac{-\left[\sin\left(\log x\right)+\cos\left(\log x\right)\right]}{x^2}$$

Question 11:

If $y=5\cos x-3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$

Answer

It is given that, $y=5\cos x-3\sin x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(5\cos x)-\frac{d}{dx}(3\sin x)=5\frac{d}{dx}(\cos x)-3\frac{d}{dx}(\sin x)$$
$$=5(-\sin x)-3\cos x=-(5\sin x+3\cos x)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-(5\sin x+3\cos x)\right]$$
$$=-\left[5\cdot\frac{d}{dx}(\sin x)+3\cdot\frac{d}{dx}(\cos x)\right]$$
$$=-\left[5\cos x+3(-\sin x)\right]$$
$$=-\left[5\cos x-3\sin x\right]$$
$$=-y$$
$$\therefore\frac{d^2y}{dx^2}+y=0$$

Hence, proved.

Question 12:

If $y=\cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer

It is given that, $y=\cos^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}=-\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$=-\left(-\frac{1}{2}\right)\cdot\left(1-x^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$=\frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times\left(-2x\right)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{\left(1-x^2\right)^3}} \qquad ...(i)$$

$y=\cos^{-1}x\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$=\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow\frac{d^2y}{dx^2}=-\cot y\cdot\operatorname{cosec}^2 y$$

Question 13:

If $y=3\cos\left(\log x\right)+4\sin\left(\log x\right)$, show that $x^2y_2+xy_1+y=0$

Answer

It is given that, $y=3\cos\left(\log x\right)+4\sin\left(\log x\right)$

Then,

$$y_1 = 3\cdot\frac{d}{dx}\left[\cos\left(\log x\right)\right] + 4\cdot\frac{d}{dx}\left[\sin\left(\log x\right)\right]$$

$$= 3\cdot\left[-\sin\left(\log x\right)\cdot\frac{d}{dx}\left(\log x\right)\right] + 4\cdot\left[\cos\left(\log x\right)\cdot\frac{d}{dx}\left(\log x\right)\right]$$

$$\therefore y_1 = \frac{-3\sin\left(\log x\right)}{x} + \frac{4\cos\left(\log x\right)}{x} = \frac{4\cos\left(\log x\right) - 3\sin\left(\log x\right)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos\left(\log x\right) - 3\sin\left(\log x\right)}{x}\right)$$

$$= \frac{x\left\{4\cos\left(\log x\right) - 3\sin\left(\log x\right)\right\}' - \left\{4\cos\left(\log x\right) - 3\sin\left(\log x\right)\right\}(x)'}{x^2}$$

$$= \frac{x\left[4\left\{\cos\left(\log x\right)\right\}' - 3\left\{\sin\left(\log x\right)\right\}'\right] - \left\{4\cos\left(\log x\right) - 3\sin\left(\log x\right)\right\}.1}{x^2}$$

$$= \frac{x\left[-4\sin\left(\log x\right).\left(\log x\right)' - 3\cos\left(\log x\right).\left(\log x\right)'\right] - 4\cos\left(\log x\right) + 3\sin\left(\log x\right)}{x^2}$$

$$= \frac{x\left[-4\sin\left(\log x\right).\frac{1}{x} - 3\cos\left(\log x\right).\frac{1}{x}\right] - 4\cos\left(\log x\right) + 3\sin\left(\log x\right)}{x^2}$$

$$= \frac{-4\sin\left(\log x\right) - 3\cos\left(\log x\right) - 4\cos\left(\log x\right) + 3\sin\left(\log x\right)}{x^2}$$

$$= \frac{-\sin\left(\log x\right) - 7\cos\left(\log x\right)}{x^2}$$

$$\therefore x^2 y_2 + x y_1 + y$$

$$= x^2\left(\frac{-\sin\left(\log x\right) - 7\cos\left(\log x\right)}{x^2}\right) + x\left(\frac{4\cos\left(\log x\right) - 3\sin\left(\log x\right)}{x}\right) + 3\cos\left(\log x\right) + 4\sin\left(\log x\right)$$

$$= -\sin\left(\log x\right) - 7\cos\left(\log x\right) + 4\cos\left(\log x\right) - 3\sin\left(\log x\right) + 3\cos\left(\log x\right) + 4\sin\left(\log x\right)$$

$$= 0$$

Hence, proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A \cdot \frac{d}{dx}\left(e^{mx}\right) + B \cdot \frac{d}{dx}\left(e^{nx}\right) = A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am \cdot \frac{d}{dx}\left(e^{mx}\right) + Bn \cdot \frac{d}{dx}\left(e^{nx}\right)$$

$$= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n) \cdot \left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Answer

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx}=500.\frac{d}{dx}\left(e^{7x}\right)+600.\frac{d}{dx}\left(e^{-7x}\right)$$
$$=500\cdot e^{7x}\cdot\frac{d}{dx}(7x)+600\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$$
$$=3500e^{7x}-4200e^{-7x}$$
$$\therefore\frac{d^2y}{dx^2}=3500\cdot\frac{d}{dx}\left(e^{7x}\right)-4200.\frac{d}{dx}\left(e^{-7x}\right)$$
$$=3500\cdot e^{7x}.\frac{d}{dx}(7x)-4200\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$$
$$=7\times3500\cdot e^{7x}+7\times4200\cdot e^{-7x}$$
$$=49\times500e^{7x}+49\times600e^{-7x}$$
$$=49\left(500e^{7x}+600e^{-7x}\right)$$
$$=49y$$

Hence, proved.

Question 16:

If $e^{y}(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

Answer

The given relationship is $e^{y}(x+1)=1$

$$e^{y}(x+1)=1$$
$$\Rightarrow e^{y}=\frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y=\log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1).\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Question 17:

If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2$

Answer

The given relationship is $y=\left(\tan^{-1}x\right)^2$

Then,

$$y_1=2\tan^{-1}x\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\Rightarrow y_1=2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1+x^2\right)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2 y_2+2x\left(1+x^2\right)y_1=2$$

Hence, proved.

Exercise 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$ Answer

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

$f(-4) = (-4)^2 + 2 \times (-4) - 8 = 16 - 8 - 8 = 0$

$f(2) = (2)^2 + 2 \times 2 - 8 = 4 + 4 - 8 = 0$

$\therefore f(-4) = f(2) = 0$

$\Rightarrow$ The value of f (x) at −4 and 2 coincides.

Rolle's Theorem states that there is a point c ∈ (−4, 2) such that $f'(c) = 0$

$f(x) = x^2 + 2x - 8$

$\Rightarrow f'(x) = 2x + 2$

$\therefore f'(c) = 0$

$\Rightarrow 2c + 2 = 0$

$\Rightarrow c = -1,$ where $c = -1 \in (-4, 2)$

Hence, Rolle's Theorem is verified for the given function.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

Answer

By Rolle's Theorem, for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) f (a) = f (b)

then, there exists some c ∈ (a, b) such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function f (x) is not continuous at every integral point. In particular, f(x) is not continuous at x = 5 and x = 9

⇒ f (x) is not continuous in [5, 9].

$\text{Also, } f(5)=[5]=5 \text{ and } f(9)=[9]=9$

$\therefore f(5)\neq f(9)$

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that n ∈ (5, 9).

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function f (x) is not continuous at every integral point. In particular, f(x) is not continuous at x = −2 and x = 2

⇒ f (x) is not continuous in [−2, 2].

Also, $f(-2)=[-2]=-2$ and $f(2)=[2]=2$

$\therefore f(-2)\neq f(2)$

The differentiability of f in (−2, 2) is checked as follows.

Let n be an integer such that n ∈ (−2, 2).

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (−2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$f(1)=(1)^2-1=0$

$f(2)=(2)^2-1=3$

∴f (1) ≠ f (2)

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1 \text{ for } x \in [1, 2]$.

Question 3:

If $f : [-5,5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Answer

It is given that $f : [-5,5] \to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on [−5, 5].

(b) f is differentiable on (−5, 5).

Therefore, by the Mean Value Theorem, there exists c ∈ (−5, 5) such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10 f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c) \neq 0$$

$$\Rightarrow 10 f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Answer

The given function is $f(x)=x^2-4x-3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is 2x − 4.

$$f(1)=1^2-4\times1-3=-6,\ f(4)=4^2-4\times4-3=-3$$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{-3-(-6)}{3}=\frac{3}{3}=1 \qquad f'(c)=1$$

Mean Value Theorem states that there is a point c ∈ (1, 4) such that

$$f'(c)=1$$

$$\Rightarrow 2c-4=1$$

$$\Rightarrow c=\frac{5}{2}, \text{ where } c=\frac{5}{2}\in(1,\ 4)$$

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x)=x^3-5x^2-3x$ in the interval [a, b], where a = 1 and b = 3. Find all $c\in(1,3)$ for which $f'(c)=0$ Answer

The given function f is $f(x)=x^3-5x^2-3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1)=1^3-5\times1^2-3\times1=-7,\ f(3)=3^3-5\times3^2-3\times3=-27$$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{-27-(-7)}{3-1}=-10$$

Mean Value Theorem states that there exist a point c ∈ (1, 3) such that $f'(c) = -10$

$$f'(c) = -10$$
$$\Rightarrow 3c^2 - 10c - 3 = 10$$
$$\Rightarrow 3c^2 - 10c + 7 = 0$$
$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$
$$\Rightarrow 3c(c-1) - 7(c-1) = 0$$
$$\Rightarrow (c-1)(3c-7) = 0$$
$$\Rightarrow c = 1,\ \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1,\ 3)$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1,\ 3)$ is the only point for which $f'(c) = 0$

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer

Mean Value Theorem states that for a function $f : [a,\ b] \to \mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

then, there exists some c ∈ (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function f (x) is not continuous at every integral point. In particular, f(x) is not continuous at x = 5 and x = 9

$\Rightarrow$ f (x) is not continuous in [5, 9].

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that n $\in$ (5, 9).

The left hand limit of f at $x=n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

$\therefore$f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = −2 and x = 2
⇒ f (x) is not continuous in [−2, 2].

The differentiability of f in (−2, 2) is checked as follows.
Let n be an integer such that n ∈ (−2, 2).

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (−2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable

for $f(x)=x^2-1 \text{ for } x\in[1,\ 2]$. It can be proved as follows.

$f(1)=1^2-1=0,\ f(2)=2^2-1=3$

$\therefore \dfrac{f(b)-f(a)}{b-a}=\dfrac{f(2)-f(1)}{2-1}=\dfrac{3-0}{1}=3$

$f'(x)=2x$

$\therefore f'(c)=3$

$\Rightarrow 2c=3$

$\Rightarrow c=\dfrac{3}{2}=1.5,\ \text{where } 1.5\in[1,2]$

Miscellaneous Solutions

Question 1:

$\left(3x^2-9x+5\right)^9$

Answer

Let $y=\left(3x^2-9x+5\right)^9$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(3x^2-9x+5\right)^9$$
$$=9\left(3x^2-9x+5\right)^8\cdot\frac{d}{dx}\left(3x^2-9x+5\right)$$
$$=9\left(3x^2-9x+5\right)^8\cdot\left(6x-9\right)$$
$$=9\left(3x^2-9x+5\right)^8\cdot 3\left(2x-3\right)$$
$$=27\left(3x^2-9x+5\right)^8\left(2x-3\right)$$

Question 2:

$\sin^3 x+\cos^6 x$

Answer

Let $y=\sin^3 x+\cos^6 x$

$$\therefore\frac{dy}{dx}=\frac{d}{dx}\left(\sin^3 x\right)+\frac{d}{dx}\left(\cos^6 x\right)$$
$$=3\sin^2 x\cdot\frac{d}{dx}\left(\sin x\right)+6\cos^5 x\cdot\frac{d}{dx}\left(\cos x\right)$$
$$=3\sin^2 x\cdot\cos x+6\cos^5 x\cdot\left(-\sin x\right)$$
$$=3\sin x\cos x\left(\sin x-2\cos^4 x\right)$$

Question 3:

$\left(5x\right)^{3\cos 2x}$

Answer

$\text{Let } y = (5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y = 3\cos 2x \log 5x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\log 5x(-\sin 2x)\cdot \frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x}\cdot \frac{d}{dx}(5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[-2\sin 2x \log 5x + \frac{\cos 2x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

$$\therefore \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

Question 4:

$\sin^{-1}\left(x\sqrt{x}\right),\ 0 \le x \le 1$

Answer

$\text{Let } y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$
$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} \cdot x^{\frac{1}{2}}$$
$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$
$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5:

$$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, \quad -2 < x < 2$$

Answer

Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{2}\right)\right] - \left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{2x+7}$$

$$= \frac{\sqrt{2x+7}\,\frac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)} - \frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{}{2}$$

Answer

$$\text{Let } y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] \quad \ldots(1)$$

$$\text{Then, } \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$$

$$= \frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$= \frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$= \frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$= \frac{1+\cos x}{\sin x}$$

$$= \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y = \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y = \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

Question 7:

$\left(\log x\right)^{\log x}$, $x > 1$

Answer

$$\text{Let } y = \left(\log x\right)^{\log x}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log\left(\log x\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log\left(\log x\right)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log\left(\log x\right).\frac{d}{dx}\left(\log x\right) + \log x \cdot \frac{d}{dx}\left[\log\left(\log x\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log\left(\log x\right).\frac{1}{x} + \log x \cdot \frac{1}{\log x}\cdot\frac{d}{dx}\left(\log x\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log\left(\log x\right) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = \left(\log x\right)^{\log x}\left[\frac{1}{x} + \frac{\log\left(\log x\right)}{x}\right]$$

Question 8:

$\cos\left(a\cos x + b\sin x\right)$, for some constant a and b.

Answer

$$\text{Let } y = \cos\left(a\cos x + b\sin x\right)$$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left(a\cos x + b\sin x\right)$$

$$\Rightarrow \frac{dy}{dx} = -\sin\left(a\cos x + b\sin x\right)\cdot\frac{d}{dx}\left(a\cos x + b\sin x\right)$$

$$= -\sin\left(a\cos x + b\sin x\right)\cdot\left[a\left(-\sin x\right) + b\cos x\right]$$

$$= \left(a\sin x - b\cos x\right)\cdot\sin\left(a\cos x + b\sin x\right)$$

Question 9:

$\left(\sin x - \cos x\right)^{\left(\sin x - \cos x\right)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer

$\text{Let } y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking logarithm on both the sides, we obtain

$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$

$\Rightarrow \log y = (\sin x - \cos x)\cdot \log(\sin x - \cos x)$

Differentiating both sides with respect to x, we obtain

$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x)\log(\sin x - \cos x)\right]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x)\cdot \frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x)\cdot \frac{d}{dx}\log(\sin x - \cos x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x)\cdot (\cos x + \sin x) + (\sin x - \cos x)\cdot \frac{1}{(\sin x - \cos x)}\cdot \frac{d}{dx}(\sin x - \cos x)$

$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\left[(\cos x + \sin x)\cdot \log(\sin x - \cos x) + (\cos x + \sin x)\right]$

$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right]$

Question 10:

$x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

Answer

$\text{Let } y = x^x + x^a + a^x + a^a$

$\text{Also, let } x^x = u,\ x^a = v,\ a^x = w, \text{ and } a^a = s$

$\therefore y = u + v + w + s$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \qquad \ldots(1)$

$u = x^x$

$\Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \cdot 1 + x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x\left[\log x + 1\right] = x^x\left(1 + \log x\right) \qquad \ldots(2)$$

$$v = x^a$$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}\left(x^a\right)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \qquad \ldots(3)$$

$$w = a^x$$

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \qquad \ldots(4)$$

$s = a^a$

Since a is constant, a^a is also a constant.

$$\therefore \frac{ds}{dx} = 0 \qquad \ldots(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx} = x^x\left(1 + \log x\right) + ax^{a-1} + a^x \log a + 0$$

$$= x^x\left(1 + \log x\right) + ax^{a-1} + a^x \log a$$

Question 11:

$x^{x^2-3}+(x-3)^{x^2}$, for $x>3$

Answer

Let $y=x^{x^2-3}+(x-3)^{x^2}$

Also, let $u=x^{x^2-3}$ and $v=(x-3)^{x^2}$

$\therefore y=u+v$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=x^{x^2-3}$

$\therefore \log u=\log\left(x^{x^2-3}\right)$

$\log u=\left(x^2-3\right)\log x$

Differentiating with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\log x\cdot\frac{d}{dx}\left(x^2-3\right)+\left(x^2-3\right)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=\log x\cdot 2x+\left(x^2-3\right)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{du}{dx}=x^{x^2-3}\cdot\left[\frac{x^2-3}{x}+2x\log x\right]$$

Also,

$v=(x-3)^{x^2}$

$\therefore \log v=\log(x-3)^{x^2}$

$\Rightarrow \log v=x^2\log(x-3)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\log(x-3)\cdot\frac{d}{dx}(x^2)+x^2\cdot\frac{d}{dx}\left[\log(x-3)\right]$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\log(x-3)\cdot 2x+x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$

$$\Rightarrow\frac{dv}{dx}=v\left[2x\log(x-3)+\frac{x^2}{x-3}\cdot 1\right]$$

$$\Rightarrow\frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y=12(1-\cos t),x=10(t-\sin t),-\frac{\pi}{2}<t<\frac{\pi}{2}$

Answer

It is given that, $y=12(1-\cos t),x=10(t-\sin t)$

$$\therefore\frac{dx}{dt}=\frac{d}{dt}\left[10(t-\sin t)\right]=10\cdot\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[12(1-\cos t)\right]=12\cdot\frac{d}{dt}(1-\cos t)=12\cdot\left[0-(-\sin t)\right]=12\sin t$$

$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10(1-\cos t)}=\frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},\ -1\le x\le 1$

Answer

It is given that, $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1} x + \sin^{-1} \sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x\right) + \frac{d}{dx}\left(\sin^{-1} \sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}}\left(-2x\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = 0$$

Question 14:

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Answer

It is given that,

$x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we obtain

$x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 + x^2y = y^2 + xy^2$

$\Rightarrow x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow x^2 - y^2 = xy(y-x)$

$\Rightarrow (x+y)(x-y) = xy(y-x)$

$\therefore x + y = -xy$

$\Rightarrow (1+x)y = -x$

$\Rightarrow y = \frac{-x}{(1+x)}$

Differentiating both sides with respect to x, we obtain

$y = \frac{-x}{(1+x)}$

$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$

Hence, proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

Answer

It is given that, $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= -\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$= -c,$ which is constant and is independent of a and b

Hence, proved.

Question 16:

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Answer

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \qquad ...(1)$$

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)]\cdot\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Answer

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a\cdot\frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t\cdot\frac{d}{dx}(t) + t\cdot\frac{d}{dt}(\sin t)\right]$$

$$= a\left[-\sin t + \sin t + t\cos t\right] = at\cos t$$

$$\frac{dy}{dt} = a\cdot\frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t\cdot\frac{d}{dt}(t) + t\cdot\frac{d}{dt}(\cos t)\right\}\right]$$

$$= a\left[\cos t - \left\{\cos t - t\sin t\right\}\right] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t\cdot\frac{dt}{dx}$

$$= \sec^2 t\cdot\frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x, and find it.

Answer

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when x ≥ 0, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When x < 0, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n. Answer

To prove: $P(n): \frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For n = 1,

$$P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

∴P(n) is true for n = 1

Let P(k) is true for some positive integer k.

That is, $P(k): \frac{d}{dx}(x^k) = kx^{k-1}$

It has to be proved that P(k + 1) is also true.

$$\text{Consider } \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k)$$

$$= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \qquad [\text{By applying product rule}]$$

$$= x^k \cdot 1 + x \cdot k \cdot x^{k-1}$$

$$= x^k + kx^k$$

$$= (k+1) \cdot x^k$$

$$= (k+1) \cdot x^{(k+1)-1}$$

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Answer

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\frac{d}{dx}(\sin A)+\sin A\cdot\frac{d}{dx}(\cos B)$$

$$+\sin B\cdot\frac{d}{dx}(\cos A)+\cos A\cdot\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\cos A\frac{dA}{dx}+\sin A(-\sin B)\frac{dB}{dx}$$

$$+\sin B(-\sin A)\cdot\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$

$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Question 22:

If $y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer

$$y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

$$\text{Then, } \frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $$\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Question 23:

If $y=e^{a\cos^{-1}x}, -1\le x\le 1$, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Answer

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both the sides, we obtain

$\log\ y = a\cos^{-1} x \log e$

$\log\ y = a\cos^{-1} x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(1-x^2\right) + \left(1-x^2\right) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2 \frac{d}{dx}\left(y^2\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 \left(-2x\right) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 \left(-2x\right) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = a^2.y \qquad \left[\frac{dy}{dx} \neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence, proved.