Exercise 7.1

Question 1:

sin 2x

Answer

The anti derivative of sin 2x is a function of x whose derivative is sin 2x.It is known that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Therefore, the anti derivative of $\sin 2x$ is $-\frac{1}{2}\cos 2x$

Question 2:

Cos 3x

Answer

The anti derivative of cos 3x is a function of x whose derivative is cos 3x. It is known that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Therefore, the anti derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$.

Question 3:

e^{2x}

Answer

The anti derivative of e^{2x} is the function of x whose derivative is e^{2x}. It is known that,

$$\frac{d}{dx}\left(e^{2x}\right)=2e^{2x}$$
$$\Rightarrow e^{2x}=\frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$
$$\therefore e^{2x}=\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Question 4:

$(ax+b)^2$

Answer

The anti derivative of $(ax+b)^2$ is the function of x whose derivative is $(ax+b)^2$. It is known that,

$$\frac{d}{dx}(ax+b)^3=3a(ax+b)^2$$
$$\Rightarrow (ax+b)^2=\frac{1}{3a}\frac{d}{dx}(ax+b)^3$$
$$\therefore (ax+b)^2=\frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Therefore, the anti derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

Question 5:

$\sin 2x-4e^{3x}$

Answer

The anti derivative of $\left(\sin 2x-4e^{3x}\right)$ is the function of x whose derivative is $\left(\sin 2x-4e^{3x}\right)$.

It is known that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)=\sin 2x-4e^{3x}$$

Therefore, the anti derivative of $\left(\sin 2x-4e^{3x}\right)$ is $\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)$.

Question 6:

$$\int\left(4e^{3x}+1\right)dx$$

Answer

$$\int\left(4e^{3x}+1\right)dx$$

$$=4\int e^{3x}dx+\int 1dx$$

$$=4\left(\frac{e^{3x}}{3}\right)+x+\mathrm{C}$$

$$=\frac{4}{3}e^{3x}+x+\mathrm{C}$$

Question 7:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

Answer

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

$$=\int\left(x^2-1\right)dx$$

$$=\int x^2dx-\int 1dx$$

$$=\frac{x^3}{3}-x+\mathrm{C}$$

Question 8:

$$\int\left(ax^2+bx+c\right)dx$$

Answer

$$\int\left(ax^2+bx+c\right)dx$$
$$=a\int x^2dx+b\int xdx+c\int 1.dx$$
$$=a\left(\frac{x^3}{3}\right)+b\left(\frac{x^2}{2}\right)+cx+C$$
$$=\frac{ax^3}{3}+\frac{bx^2}{2}+cx+C$$

Question 9:

$$\int\left(2x^2+e^x\right)dx$$

Answer

$$\int\left(2x^2+e^x\right)dx$$
$$=2\int x^2dx+\int e^xdx$$
$$=2\left(\frac{x^3}{3}\right)+e^x+C$$
$$=\frac{2}{3}x^3+e^x+C$$

Question 10:

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$$

Answer

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$$
$$=\int\left(x+\frac{1}{x}-2\right)dx$$
$$=\int xdx+\int\frac{1}{x}dx-2\int 1.dx$$
$$=\frac{x^2}{2}+\log|x|-2x+C$$

Question 11:

$$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$$

Answer

$$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$$

$$= \int \left(x + 5 - 4x^{-2}\right) dx$$

$$= \int x dx + 5 \int 1.dx - 4 \int x^{-2} dx$$

$$= \frac{x^2}{2} + 5x - 4\left(\frac{x^{-1}}{-1}\right) + C$$

$$= \frac{x^2}{2} + 5x + \frac{4}{x} + C$$

Question 12:

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx$$

Answer

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx$$

$$= \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \right) dx$$

$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + \frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}} + C$$

$$= \frac{2}{7} x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$$

$$= \frac{2}{7} x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C$$

Question 13:

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

Answer

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

On dividing, we obtain

$$= \int (x^2 + 1) dx$$

$$= \int x^2 dx + \int 1 dx$$

$$= \frac{x^3}{3} + x + C$$

Question 14:

$$\int (1 - x)\sqrt{x} dx$$

Answer

$$\int (1 - x)\sqrt{x} dx$$

$$= \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx$$

$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$$

Question 15:

$$\int \sqrt{x} \left(3x^2 + 2x + 3 \right) dx$$

Answer

$$\int \sqrt{x}\left(3x^2+2x+3\right)dx$$

$$= \int\left(3x^{\frac{5}{2}}+2x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx$$

$$= 3\int x^{\frac{5}{2}}dx+2\int x^{\frac{3}{2}}dx+3\int x^{\frac{1}{2}}dx$$

$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right)+2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)+3\frac{\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+\text{C}$$

$$= \frac{6}{7}x^{\frac{7}{2}}+\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+\text{C}$$

Question 16:

$$\int\left(2x-3\cos x+e^x\right)dx$$

Answer

$$\int\left(2x-3\cos x+e^x\right)dx$$

$$= 2\int x dx-3\int \cos x dx+\int e^x dx$$

$$= \frac{2x^2}{2}-3\left(\sin x\right)+e^x+\text{C}$$

$$= x^2-3\sin x+e^x+\text{C}$$

Question 17:

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$

Answer

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$

$$= 2\int x^2 dx - 3\int \sin x dx + 5\int x^{\frac{1}{2}} dx$$

$$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$$

Question 18:

$$\int \sec x(\sec x + \tan x) dx$$

Answer

$$\int \sec x(\sec x + \tan x) dx$$

$$= \int (\sec^2 x + \sec x \tan x) dx$$

$$= \int \sec^2 x dx + \int \sec x \tan x dx$$

$$= \tan x + \sec x + C$$

Question 19:

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$$

Answer

$$\int \frac{\sec^2 x}{\cos ec^2 x} dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$= \int \tan^2 x dx$$

$$= \int \left(\sec^2 x - 1\right) dx$$

$$= \int \sec^2 x dx - \int 1 dx$$

$$= \tan x - x + C$$

Question 20:

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$

Answer

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$$

$$= \int 2\sec^2 x dx - 3\int \tan x \sec x dx$$

$$= 2\tan x - 3\sec x + C$$

Question 21:

The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

Answer

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$

$$=\int x^{\frac{1}{2}}dx+\int x^{-\frac{1}{2}}dx$$

$$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+C$$

$$=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$$

Hence, the correct Answer is C.

Question 22:

If $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$ such that f(2) = 0, then f(x) is

(A) $x^4+\frac{1}{x^3}-\frac{129}{8}$ (B) $x^3+\frac{1}{x^4}+\frac{129}{8}$

(C) $x^4+\frac{1}{x^3}+\frac{129}{8}$ (D) $x^3+\frac{1}{x^4}-\frac{129}{8}$

Answer

It is given that,

$\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$

∴Anti derivative of $4x^3-\frac{3}{x^4}=f(x)$

$$\therefore f(x) = \int 4x^3 - \frac{3}{x^4}\,dx$$

$$f(x) = 4\int x^3 dx - 3\int (x^{-4})dx$$

$$\therefore f(x) = 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C$$

$$f(x) = x^4 + \frac{1}{x^3} + C$$

Also,

$$f(2) = 0$$

$$\therefore f(2) = (2)^4 + \frac{1}{(2)^3} + C = 0$$

$$\Rightarrow 16 + \frac{1}{8} + C = 0$$

$$\Rightarrow C = -\left(16 + \frac{1}{8}\right)$$

$$\Rightarrow C = \frac{-129}{8}$$

$$\therefore f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$$

Hence, the correct Answer is A.

Exercise 7.2

Question 1:

$$\frac{2x}{1+x^2}$$

Answer

Let $1+x^2$ = t

∴2x dx = dt

$$\Rightarrow \int \frac{2x}{1+x^2}dx = \int \frac{1}{t}dt$$

$$= \log|t| + C$$

$$= \log|1+x^2| + C$$

$$= \log(1+x^2) + C$$

Question 2:

$$\frac{(\log x)^2}{x}$$

Answer

Let log |x| = t

$$\therefore \frac{1}{x}dx = dt$$

$$\Rightarrow \int \frac{(\log|x|)^2}{x}dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(\log|x|)^3}{3} + C$$

Question 3:

$$\frac{1}{x + x\log x}$$

Answer

$$\frac{1}{x + x\log x} = \frac{1}{x(1+\log x)}$$

Let 1 + log x = t

$$\therefore \frac{1}{x}dx = dt$$

$$\Rightarrow \int \frac{1}{x(1+\log x)}dx = \int \frac{1}{t}dt$$

$$= \log|t| + C$$

$$= \log|1+\log x| + C$$

Question 4:

sin x · sin (cos x)

Answer

sin x · sin (cos x)

Let cos x = t

∴ −sin x dx = dt

$$\Rightarrow \int \sin x \cdot \sin(\cos x)\, dx = -\int \sin t\, dt$$
$$= -[-\cos t] + C$$
$$= \cos t + C$$
$$= \cos(\cos x) + C$$

Question 5:

$\sin(ax+b)\cos(ax+b)$

Answer

$$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$$

Let $2(ax+b) = t$

∴ 2adx = dt

$$\Rightarrow \int \frac{\sin 2(ax+b)}{2} dx = \frac{1}{2}\int \frac{\sin t\ dt}{2a}$$
$$= \frac{1}{4a}[-\cos t] + C$$
$$= \frac{-1}{4a}\cos 2(ax+b) + C$$

Question 6:

$\sqrt{ax+b}$

Answer

Let ax + b = t

⇒ adx = dt

$$\therefore dx = \frac{1}{a}dt$$

$$\Rightarrow \int (ax+b)^{\frac{1}{2}} dx = \frac{1}{a}\int t^{\frac{1}{2}} dt$$

$$= \frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$$

Question 7:

$x\sqrt{x+2}$

Answer

Let $(x+2) = t$

∴ dx = dt

$$\Rightarrow \int x\sqrt{x+2}\,dx = \int (t-2)\sqrt{t}\,dt$$
$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right) dt$$
$$= \int t^{\frac{3}{2}} dt - 2\int t^{\frac{1}{2}} dt$$
$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$
$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Question 8:

$x\sqrt{1+2x^2}$

Answer

Let $1 + 2x^2 = t$

∴ 4xdx = dt

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}\,dt}{4}$$
$$= \frac{1}{4}\int t^{\frac{1}{2}}\,dt$$
$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

Question 9:

$(4x+2)\sqrt{x^2+x+1}$

Answer

Let $x^2+x+1=t$

$\therefore (2x+1)dx = dt$

$$\int (4x+2)\sqrt{x^2+x+1}\;dx$$
$$= \int 2\sqrt{t}\;dt$$
$$= 2\int \sqrt{t}\;dt$$
$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

Question 10:

$$\frac{1}{x-\sqrt{x}}$$

Answer

$$\frac{1}{x-\sqrt{x}}=\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right)=t$

∴

$$\frac{1}{2\sqrt{x}}dx=dt$$

$$\Rightarrow \int\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx=\int\frac{2}{t}dt$$

$$=2\log\left|t\right|+C$$

$$=2\log\left|\sqrt{x}-1\right|+C$$

Question 11:

$$\frac{x}{\sqrt{x+4}},x>0$$

Answer

Let $x+4=t$

∴ dx = dt

$$\int \frac{x}{\sqrt{x+4}}\,dx = \int \frac{(t-4)}{\sqrt{t}}dt$$
$$= \int\left(\sqrt{t} - \frac{4}{\sqrt{t}}\right)dt$$
$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + \mathrm{C}$$
$$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + \mathrm{C}$$
$$= \frac{2}{3}t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + \mathrm{C}$$
$$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + \mathrm{C}$$
$$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12) + \mathrm{C}$$
$$= \frac{2}{3}\sqrt{x+4}\,(x-8) + \mathrm{C}$$

Question 12:

$(x^3-1)^{\frac{1}{3}} x^5$

Answer

Let $x^3 - 1 = t$

$\therefore 3x^2 dx = dt$

$$\Rightarrow \int\left(x^3-1\right)^{\frac{1}{3}} x^5 dx = \int\left(x^3-1\right)^{\frac{1}{3}} x^3 \cdot x^2 dx$$

$$= \int t^{\frac{1}{3}} (t+1)\frac{dt}{3}$$

$$= \frac{1}{3}\int\left(t^{\frac{4}{3}} + t^{\frac{1}{3}}\right)dt$$

$$= \frac{1}{3}\left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right] + C$$

$$= \frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}} + \frac{3}{4}t^{\frac{4}{3}}\right] + C$$

$$= \frac{1}{7}\left(x^3-1\right)^{\frac{7}{3}} + \frac{1}{4}\left(x^3-1\right)^{\frac{4}{3}} + C$$

Question 13:

$$\frac{x^2}{\left(2+3x^3\right)^3}$$

Answer

Let $2+3x^3 = t$

$\therefore 9x^2$

dx = dt

$$\Rightarrow \int \frac{x^2}{\left(2+3x^3\right)^3}dx = \frac{1}{9}\int \frac{dt}{(t)^3}$$

$$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right] + C$$

$$= \frac{-1}{18}\left(\frac{1}{t^2}\right) + C$$

$$= \frac{-1}{18\left(2+3x^3\right)^2} + C$$

Question 14:

$\frac{1}{x(\log x)^m}$, $x > 0$

Answer

Let log x = t

$\therefore \frac{1}{x}dx = dt$

$$\Rightarrow \int \frac{1}{x(\log x)^m}dx = \int \frac{dt}{(t)^m}$$

$$= \left(\frac{t^{-m+1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

Question 15:

$\frac{x}{9-4x^2}$

Answer

Let $9-4x^2=t$
∴ −8x dx = dt

$$\Rightarrow \int \frac{x}{9-4x^2}\,dx = \frac{-1}{8}\int \frac{1}{t}\,dt$$
$$= \frac{-1}{8}\log|t| + C$$
$$= \frac{-1}{8}\log\left|9-4x^2\right| + C$$

Question 16:

e^{2x+3}

Answer

Let $2x+3=t$
∴ 2dx = dt

$$\Rightarrow \int e^{2x+3}\,dx = \frac{1}{2}\int e^t\,dt$$
$$= \frac{1}{2}\left(e^t\right) + C$$
$$= \frac{1}{2}e^{(2x+3)} + C$$

Question 17:

$$\frac{x}{e^{x^2}}$$

Answer

Let $x^2 = t$

∴ 2xdx = dt

$$\Rightarrow \int \frac{x}{e^{x^2}} dx = \frac{1}{2} \int \frac{1}{e^t} dt$$

$$= \frac{1}{2} \int e^{-t} dt$$

$$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + \mathrm{C}$$

$$= -\frac{1}{2} e^{-x^2} + \mathrm{C}$$

$$= \frac{-1}{2e^{x^2}} + \mathrm{C}$$

Question 18:

$$\frac{e^{\tan^{-1} x}}{1+x^2}$$

Answer

Let $\tan^{-1} x = t$

∴ $\frac{1}{1+x^2} dx = dt$

$$\Rightarrow \int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^t dt$$

$$= e^t + C$$

$$= e^{\tan^{-1} x} + C$$

Question 19:

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Answer

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Dividing numerator and denominator by e^x, we obtain

$$\frac{\frac{\left(e^{2x}-1\right)}{e^x}}{\frac{\left(e^{2x}+1\right)}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

$\therefore \left(e^x - e^{-x}\right) dx = dt$

$$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$= \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^x + e^{-x}\right| + C$$

Question 20:

$$\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$$

Answer

Let $e^{2x}+e^{-2x}=t$

$\therefore \left(2e^{2x}-2e^{-2x}\right)dx = dt$

$$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx = dt$$

$$\Rightarrow \int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx = \int\frac{dt}{2t}$$

$$=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+\text{C}$$

$$=\frac{1}{2}\log\left|e^{2x}+e^{-2x}\right|+\text{C}$$

Question 21:

$$\tan^2(2x-3)$$

Answer

$$\tan^2(2x-3)=\sec^2(2x-3)-1$$

Let $2x - 3 = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \tan^2(2x-3)\,dx = \int\left[\left(\sec^2(2x-3)\right)-1\right]dx$$
$$= \frac{1}{2}\int\left(\sec^2 t\right)dt - \int 1dx$$
$$= \frac{1}{2}\int \sec^2 t\,dt - \int 1dx$$
$$= \frac{1}{2}\tan t - x + C$$
$$= \frac{1}{2}\tan(2x-3) - x + C$$

Question 22:

$\sec^2(7-4x)$

Answer

Let 7 − 4x = t

∴ −4dx = dt

$$\therefore \int \sec^2(7-4x)\,dx = \frac{-1}{4}\int \sec^2 t\,dt$$
$$= \frac{-1}{4}(\tan t) + C$$
$$= \frac{-1}{4}\tan(7-4x) + C$$

Question 23:

$\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

Answer

Let $\sin^{-1} x = t$

∴

$$\frac{1}{\sqrt{1-x^2}}dx = dt$$

$$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx = \int t\,dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{\left(\sin^{-1} x\right)^2}{2} + C$$

Question 24:

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$$

Answer

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2\left(3\cos x + 2\sin x\right)}$$

Let $3\cos x + 2\sin x = t$

∴ $\left(-3\sin x + 2\cos x\right)dx = dt$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\,dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t}dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + C$$

Question 25:

$$\frac{1}{\cos^2 x\left(1-\tan x\right)^2}$$

Answer

$$\frac{1}{\cos^2 x\left(1-\tan x\right)^2}=\frac{\sec^2 x}{\left(1-\tan x\right)^2}$$

Let $\left(1-\tan x\right)=t$

$\therefore -\sec^2 x dx = dt$

$$\Rightarrow \int\frac{\sec^2 x}{\left(1-\tan x\right)^2}dx=\int\frac{-dt}{t^2}$$

$$=-\int t^{-2}dt$$

$$=+\frac{1}{t}+\mathrm{C}$$

$$=\frac{1}{\left(1-\tan x\right)}+\mathrm{C}$$

Question 26:

$$\frac{\cos\sqrt{x}}{\sqrt{x}}$$

Answer

Let $\sqrt{x}=t$

$\therefore \frac{1}{2\sqrt{x}}dx=dt$

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos t\,dt$$
$$= 2\sin t + C$$
$$= 2\sin\sqrt{x} + C$$

Question 27:

$\sqrt{\sin 2x}\cos 2x$

Answer

Let sin 2x = t

$\therefore 2\cos 2x\,dx = dt$

$$\Rightarrow \int \sqrt{\sin 2x}\ \cos 2x\,dx = \frac{1}{2}\int \sqrt{t}\,dt$$
$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{3}t^{\frac{3}{2}} + C$$
$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

Question 28:

$\frac{\cos x}{\sqrt{1+\sin x}}$

Answer

Let $1 + \sin x = t$

$\therefore \cos x\ dx = dt$

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{1+\sin x} + C$$

Question 29:

cot x log sin x

Answer

Let log sin x = t

$$\Rightarrow \frac{1}{\sin x} \cdot \cos x\ dx = dt$$

$$\therefore \cot x\ dx = dt$$

$$\Rightarrow \int \cot x\ \log \sin x\ dx = \int t\, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{1}{2}(\log \sin x)^2 + C$$

Question 30:

$$\frac{\sin x}{1+\cos x}$$

Answer

Let 1 + cos x = t

∴ −sin x dx = dt

$$\Rightarrow \int \frac{\sin x}{1+\cos x}\,dx = \int -\frac{dt}{t}$$
$$= -\log|t| + C$$
$$= -\log|1+\cos x| + C$$

Question 31:

$$\frac{\sin x}{(1+\cos x)^2}$$

Answer

Let 1 + cos x = t
∴ −sin x dx = dt

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2}\,dx = \int -\frac{dt}{t^2}$$
$$= -\int t^{-2}dt$$
$$= \frac{1}{t} + C$$
$$= \frac{1}{1+\cos x} + C$$

Question 32:

$$\frac{1}{1+\cot x}$$

Answer

$$\text{Let } I = \int \frac{1}{1+\cot x}dx$$

$$= \int \frac{1}{1+\frac{\cos x}{\sin x}}dx$$

$$= \int \frac{\sin x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)}dx$$

$$= \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}(x) + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x + \cos x}dx$$

Let $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\, dx = dt$

$$\therefore I = \frac{x}{2} + \frac{1}{2}\int \frac{-(dt)}{t}$$

$$= \frac{x}{2} - \frac{1}{2}\log|t| + \text{C}$$

$$= \frac{x}{2} - \frac{1}{2}\log|\sin x + \cos x| + \text{C}$$

Question 33:

$$\frac{1}{1-\tan x}$$

Answer

$$\text{Let } I = \int \frac{1}{1-\tan x}dx$$
$$= \int \frac{1}{1-\frac{\sin x}{\cos x}}dx$$
$$= \int \frac{\cos x}{\cos x - \sin x}dx$$
$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}dx$$
$$= \frac{1}{2}\int \frac{(\cos x - \sin x) + (\cos x + \sin x)}{(\cos x - \sin x)}dx$$
$$= \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}dx$$
$$= \frac{x}{2} + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}dx$$

Put cos x − sin x = t ⇒ (−sin x − cos x) dx = dt

$$\therefore I = \frac{x}{2} + \frac{1}{2}\int \frac{-(dt)}{t}$$
$$= \frac{x}{2} - \frac{1}{2}\log|t| + C$$
$$= \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$$

Question 34:

$$\frac{\sqrt{\tan x}}{\sin x \cos x}$$

Answer

Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

$= \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$

$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

$= \int \frac{\sec^2 x \, dx}{\sqrt{\tan x}}$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$\therefore I = \int \frac{dt}{\sqrt{t}}$

$= 2\sqrt{t} + \text{C}$

$= 2\sqrt{\tan x} + \text{C}$

Question 35:

$\frac{(1+\log x)^2}{x}$

Answer

Let 1 + log x = t

$\therefore \frac{1}{x} dx = dt$

$\Rightarrow \int \frac{(1+\log x)^2}{x} dx = \int t^2 dt$

$= \frac{t^3}{3} + \text{C}$

$= \frac{(1+\log x)^3}{3} + \text{C}$

Question 36:

$$\frac{(x+1)(x+\log x)^2}{x}$$

Answer

$$\frac{(x+1)(x+\log x)^2}{x}=\left(\frac{x+1}{x}\right)(x+\log x)^2=\left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $(x+\log x)=t$

∴

$$\left(1+\frac{1}{x}\right)dx=dt$$

$$\Rightarrow \int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx=\int t^2\,dt$$

$$=\frac{t^3}{3}+C$$

$$=\frac{1}{3}(x+\log x)^3+C$$

Question 37:

$$\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$$

Answer

Let $x^4 = t$

∴ $4x^3dx = dt$

$$\Rightarrow \int \frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8} dx = \frac{1}{4} \int \frac{\sin\left(\tan^{-1} t\right)}{1+t^2} dt \qquad ...(1)$$

Let $\tan^{-1} t = u$

$\therefore \frac{1}{1+t^2} dt = du$

From (1), we obtain

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right) dx}{1+x^8} = \frac{1}{4} \int \sin u \, du$$

$$= \frac{1}{4}(-\cos u) + C$$

$$= \frac{-1}{4} \cos\left(\tan^{-1} t\right) + C$$

$$= \frac{-1}{4} \cos\left(\tan^{-1} x^4\right) + C$$

Question 38:

$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $\left(10^x - x^{10}\right)^{-1} + C$ (D) $\log\left(10^x + x^{10}\right) + C$

Answer

Let $x^{10} + 10^x = t$

$\therefore \left(10x^9 + 10^x \log_e 10\right) dx = dt$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log\left(10^x + x^{10}\right) + C$$

Hence, the correct Answer is D.

Question 39:

$\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

A. $\tan x + \cot x + C$

B. $\tan x - \cot x + C$

C. $\tan x \cot x + C$

D. $\tan x - \cot 2x + C$

Answer

$$\text{Let } I = \int \frac{dx}{\sin^2 x \cos^2 x}$$

$$= \int \frac{1}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$

$$= \tan x - \cot x + C$$

Hence, the correct Answer is B.

Exercise 7.3

Question 1:

$\sin^2(2x+5)$

Answer

$$\sin^2(2x+5)=\frac{1-\cos 2(2x+5)}{2}=\frac{1-\cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5)\,dx=\int\frac{1-\cos(4x+10)}{2}dx$$

$$=\frac{1}{2}\int 1\,dx-\frac{1}{2}\int\cos(4x+10)\,dx$$

$$=\frac{1}{2}x-\frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right)+\mathrm{C}$$

$$=\frac{1}{2}x-\frac{1}{8}\sin(4x+10)+\mathrm{C}$$

Question 2:

$\sin 3x\cos 4x$

Answer

It is known that, $\sin A\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

$$\therefore \int\sin 3x\cos 4x\,dx=\frac{1}{2}\int\{\sin(3x+4x)+\sin(3x-4x)\}\,dx$$

$$=\frac{1}{2}\int\{\sin 7x+\sin(-x)\}\,dx$$

$$=\frac{1}{2}\int\{\sin 7x-\sin x\}\,dx$$

$$=\frac{1}{2}\int\sin 7x\,dx-\frac{1}{2}\int\sin x\,dx$$

$$=\frac{1}{2}\left(\frac{-\cos 7x}{7}\right)-\frac{1}{2}(-\cos x)+\mathrm{C}$$

$$=\frac{-\cos 7x}{14}+\frac{\cos x}{2}+\mathrm{C}$$

Question 3:

cos 2x cos 4x cos 6x

Answer

It is known that, $\cos A\cos B=\frac{1}{2}\left\{\cos(A+B)+\cos(A-B)\right\}$

$$\therefore \int \cos 2x(\cos 4x\cos 6x)\,dx=\int \cos 2x\left[\frac{1}{2}\left\{\cos(4x+6x)+\cos(4x-6x)\right\}\right]dx$$
$$=\frac{1}{2}\int\left\{\cos 2x\cos 10x+\cos 2x\cos(-2x)\right\}dx$$
$$=\frac{1}{2}\int\left\{\cos 2x\cos 10x+\cos^2 2x\right\}dx$$
$$=\frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\right\}+\left(\frac{1+\cos 4x}{2}\right)\right]dx$$
$$=\frac{1}{4}\int(\cos 12x+\cos 8x+1+\cos 4x)\,dx$$
$$=\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}\right]+C$$

Question 4:

$\sin^3(2x + 1)$

Answer

Let $I=\int\sin^3(2x+1)$

$$\Rightarrow \int\sin^3(2x+1)\,dx=\int\sin^2(2x+1)\cdot\sin(2x+1)\,dx$$
$$=\int\left(1-\cos^2(2x+1)\right)\sin(2x+1)\,dx$$

Let $\cos(2x+1)=t$

$\Rightarrow -2\sin(2x+1)\,dx=dt$

$\Rightarrow \sin(2x+1)\,dx=\frac{-dt}{2}$

$$\Rightarrow I = \frac{-1}{2}\int\left(1-t^2\right)dt$$

$$= \frac{-1}{2}\left\{t - \frac{t^3}{3}\right\}$$

$$= \frac{-1}{2}\left\{\cos(2x+1) - \frac{\cos^3(2x+1)}{3}\right\}$$

$$= \frac{-\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C$$

Question 5:

$\sin^3 x \cos^3 x$

Answer

$$\text{Let } I = \int \sin^3 x \cos^3 x \cdot dx$$

$$= \int \cos^3 x \cdot \sin^2 x \cdot \sin x \cdot dx$$

$$= \int \cos^3 x\left(1-\cos^2 x\right)\sin x \cdot dx$$

$$\text{Let } \cos x = t$$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\Rightarrow I = -\int t^3\left(1-t^2\right)dt$$

$$= -\int\left(t^3 - t^5\right)dt$$

$$= -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$$

$$= -\left\{\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right\} + C$$

$$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$$

Question 6:

$\sin x \sin 2x \sin 3x$

Answer

It is known that, $\sin A \sin B = \frac{1}{2}\left\{\cos(A-B)-\cos(A+B)\right\}$

$$\therefore \int \sin x \sin 2x \sin 3x \, dx = \int\left[\sin x \cdot \frac{1}{2}\left\{\cos(2x-3x)-\cos(2x+3x)\right\}\right] dx$$

$$= \frac{1}{2}\int\left(\sin x \cos(-x) - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int\left(\sin x \cos x - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \frac{\sin 2x}{2}\, dx - \frac{1}{2}\int \sin x \cos 5x \, dx$$

$$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int\left\{\frac{1}{2}\sin(x+5x)+\sin(x-5x)\right\} dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int\left(\sin 6x + \sin(-4x)\right) dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4}\right] + C$$

$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + C$$

$$= \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$$

Question 7:

sin 4x sin 8x

Answer

It is known that, $\sin A \sin B = \frac{1}{2}\cos(A-B)-\cos(A+B)$

$$\therefore \int \sin 4x \sin 8x \, dx = \int \left\{ \frac{1}{2} \cos(4x - 8x) - \cos(4x + 8x) \right\} dx$$
$$= \frac{1}{2} \int \left(\cos(-4x) - \cos 12x \right) dx$$
$$= \frac{1}{2} \int \left(\cos 4x - \cos 12x \right) dx$$
$$= \frac{1}{2} \left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right]$$

Question 8:

$$\frac{1 - \cos x}{1 + \cos x}$$

Answer

$$\frac{1 - \cos x}{1 + \cos x} = \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \qquad \left[2\sin^2 \frac{x}{2} = 1 - \cos x \text{ and } 2\cos^2 \frac{x}{2} = 1 + \cos x \right]$$
$$= \tan^2 \frac{x}{2}$$
$$= \left(\sec^2 \frac{x}{2} - 1 \right)$$
$$\therefore \int \frac{1 - \cos x}{1 + \cos x} dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx$$
$$= \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}} - x \right] + C$$
$$= 2 \tan \frac{x}{2} - x + C$$

Question 9:

$$\frac{\cos x}{1 + \cos x}$$

Answer

$$\frac{\cos x}{1+\cos x}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \text{ and } \cos x=2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]$$

$$\therefore \int\frac{\cos x}{1+\cos x}dx=\frac{1}{2}\int\left(1-\tan^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\int\left(1-\sec^2\frac{x}{2}+1\right)dx$$

$$=\frac{1}{2}\int\left(2-\sec^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[2x-\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]+C$$

$$=x-\tan\frac{x}{2}+C$$

Question 10:

$\sin^4 x$

Answer

$$\sin^4 x = \sin^2 x \sin^2 x$$
$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$
$$= \frac{1}{4}(1-\cos 2x)^2$$
$$= \frac{1}{4}\left[1+\cos^2 2x - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right)-2\cos 2x\right]$$
$$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$\therefore \int \sin^4 x\, dx = \frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right] dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right)-\frac{2\sin 2x}{2}\right]+C$$
$$= \frac{1}{8}\left[3x+\frac{\sin 4x}{4}-2\sin 2x\right]+C$$
$$= \frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x + C$$

Question 11:

$\cos^4 2x$

Answer

$$
\begin{aligned}
\cos^4 2x &= \left(\cos^2 2x\right)^2 \\
&= \left(\frac{1+\cos 4x}{2}\right)^2 \\
&= \frac{1}{4}\left[1+\cos^2 4x+2\cos 4x\right] \\
&= \frac{1}{4}\left[1+\left(\frac{1+\cos 8x}{2}\right)+2\cos 4x\right] \\
&= \frac{1}{4}\left[1+\frac{1}{2}+\frac{\cos 8x}{2}+2\cos 4x\right] \\
&= \frac{1}{4}\left[\frac{3}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int \cos^4 2x\, dx &= \int\left(\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right)dx \\
&= \frac{3}{8}x+\frac{\sin 8x}{64}+\frac{\sin 4x}{8}+\mathrm{C}
\end{aligned}
$$

Question 12:

$$\frac{\sin^2 x}{1+\cos x}$$

Answer

$$
\begin{aligned}
\frac{\sin^2 x}{1+\cos x} &= \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \qquad \left[\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2};\ \cos x = 2\cos^2\frac{x}{2}-1\right] \\
&= \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \\
&= 2\sin^2\frac{x}{2} \\
&= 1-\cos x
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int\frac{\sin^2 x}{1+\cos x}dx &= \int(1-\cos x)dx \\
&= x-\sin x+\mathrm{C}
\end{aligned}
$$

Question 13:

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos \alpha}$$

Answer

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos \alpha}=\frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2\alpha}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}} \qquad \left[\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$=\frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=\frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$=2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

$$=2\left[\cos(x)+\cos\alpha\right]$$

$$=2\cos x+2\cos\alpha$$

$$\therefore \int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos \alpha}dx=\int 2\cos x+2\cos\alpha$$

$$=2\left[\sin x+x\cos\alpha\right]+\mathrm{C}$$

Question 14:

$$\frac{\cos x-\sin x}{1+\sin 2x}$$

Answer

$$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{\left(\sin^2 x+\cos^2 x\right)+2\sin x\cos x}$$

$$\left[\sin^2 x+\cos^2 x=1;\ \ \sin 2x=2\sin x\cos x\right]$$

$$=\frac{\cos x-\sin x}{\left(\sin x+\cos x\right)^2}$$

Let $\sin x+\cos x=t$

$\therefore\left(\cos x-\sin x\right)dx=dt$

$$\Rightarrow\int\frac{\cos x-\sin x}{1+\sin 2x}dx=\int\frac{\cos x-\sin x}{\left(\sin x+\cos x\right)^2}dx$$

$$=\int\frac{dt}{t^2}$$

$$=\int t^{-2}dt$$

$$=-t^{-1}+\mathrm{C}$$

$$=-\frac{1}{t}+\mathrm{C}$$

$$=\frac{-1}{\sin x+\cos x}+\mathrm{C}$$

Question 15:

$\tan^3 2x\sec 2x$

Answer

$$\tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$$
$$= \left(\sec^2 2x - 1\right)\tan 2x \sec 2x$$
$$= \sec^2 2x \cdot \tan 2x \sec 2x - \tan 2x \sec 2x$$
$$\therefore \int \tan^3 2x \sec 2x \, dx = \int \sec^2 2x \tan 2x \sec 2x \, dx - \int \tan 2x \sec 2x \, dx$$
$$= \int \sec^2 2x \tan 2x \sec 2x \, dx - \frac{\sec 2x}{2} + C$$

Let $\sec 2x = t$

$\therefore 2\sec 2x \tan 2x \, dx = dt$

$$\therefore \int \tan^3 2x \sec 2x \, dx = \frac{1}{2}\int t^2 dt - \frac{\sec 2x}{2} + C$$
$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$
$$= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$$

Question 16:

$\tan^4 x$

Answer

$\tan^4 x$

$= \tan^2 x \cdot \tan^2 x$

$= \left(\sec^2 x - 1\right)\tan^2 x$

$= \sec^2 x \tan^2 x - \tan^2 x$

$= \sec^2 x \tan^2 x - \left(\sec^2 x - 1\right)$

$= \sec^2 x \tan^2 x - \sec^2 x + 1$

$$\therefore \int \tan^4 x \, dx = \int \sec^2 x \tan^2 x \, dx - \int \sec^2 x \, dx + \int 1 \cdot dx$$
$$= \int \sec^2 x \tan^2 x \, dx - \tan x + x + C \qquad \ldots(1)$$

Consider $\int \sec^2 x \tan^2 x \, dx$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$\Rightarrow \int \sec^2 x \tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

Question 17:

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$$

Answer

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$
$$= \tan x \sec x + \cot x \operatorname{cosec} x$$

$$\therefore \quad \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx = \int (\tan x \sec x + \cot x \operatorname{cosec} x)\, dx$$
$$= \sec x - \operatorname{cosec} x + C$$

Question 18:

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

Answer

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$
$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \qquad \left[\cos 2x = 1 - 2\sin^2 x\right]$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x$$

$$\therefore \quad \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\, dx = \int \sec^2 x\, dx = \tan x + C$$

Question 19:

$$\frac{1}{\sin x\cos^3 x}$$

Answer

$$\frac{1}{\sin x\cos^3 x}=\frac{\sin^2 x+\cos^2 x}{\sin x\cos^3 x}$$

$$=\frac{\sin x}{\cos^3 x}+\frac{1}{\sin x\cos x}$$

$$=\tan x\sec^2 x+\frac{1\cos^2 x}{\frac{\sin x\cos x}{\cos^2 x}}$$

$$=\tan x\sec^2 x+\frac{\sec^2 x}{\tan x}$$

$$\therefore\ \int\frac{1}{\sin x\cos^3 x}dx=\int\tan x\sec^2 x\ dx+\int\frac{\sec^2 x}{\tan x}\ dx$$

Let $\tan x=t\Rightarrow\sec^2 x\ dx=dt$

$$\Rightarrow\int\frac{1}{\sin x\cos^3 x}dx=\int t\,dt+\int\frac{1}{t}dt$$

$$=\frac{t^2}{2}+\log|t|+\mathrm{C}$$

$$=\frac{1}{2}\tan^2 x+\log|\tan x|+\mathrm{C}$$

Question 20:

$$\frac{\cos 2x}{(\cos x+\sin x)^2}$$

Answer

$$\frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x}=\frac{\cos 2x}{1+\sin 2x}$$

$$\therefore \int\frac{\cos 2x}{(\cos x+\sin x)^2}\,dx=\int\frac{\cos 2x}{(1+\sin 2x)}\,dx$$

Let $1+\sin 2x=t$

$\Rightarrow 2\cos 2x\ dx=dt$

$$\therefore \int\frac{\cos 2x}{(\cos x+\sin x)^2}dx=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log|1+\sin 2x|+C$$

$$=\frac{1}{2}\log\left|(\sin x+\cos x)^2\right|+C$$

$$=\log|\sin x+\cos x|+C$$

Question 21:

$\sin^{-1}(\cos x)$

Answer

$\sin^{-1}(\cos x)$

Let $\cos x=t$

Then, $\sin x=\sqrt{1-t^2}$

$$\Rightarrow (-\sin x)dx = dt$$

$$dx = \frac{-dt}{\sin x}$$

$$dx = \frac{-dt}{\sqrt{1-t^2}}$$

$$\therefore \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t\left(\frac{-dt}{\sqrt{1-t^2}}\right)$$

$$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}dt$$

Let $\sin^{-1} t = u$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}}dt = du$$

$$\therefore \int \sin^{-1}(\cos x)dx = \int 4du$$

$$= -\frac{u^2}{2} + C$$

$$= \frac{-(\sin^{1} t)^2}{2} + C$$

$$= \frac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \qquad \ldots(1)$$

It is known that,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$$

Substituting in equation (1), we obtain

$$\int \sin^{-1}(\cos x)\,dx = \frac{-\left[\frac{\pi}{2}-x\right]^2}{2}+C$$

$$= -\frac{1}{2}\left(\frac{\pi^2}{2}+x^2-\pi x\right)+C$$

$$= -\frac{\pi^2}{8}-\frac{x^2}{2}+\frac{1}{2}\pi x+C$$

$$= \frac{\pi x}{2}-\frac{x^2}{2}+\left(C-\frac{\pi^2}{8}\right)$$

$$= \frac{\pi x}{2}-\frac{x^2}{2}+C_1$$

Question 22:

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

Answer

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x-b)-(x-a)\right]}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\frac{\left[\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)\right]}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(a-b)}\left[\tan(x-b)-\tan(x-a)\right]$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \frac{1}{\sin(a-b)}\int\left[\tan(x-b)-\tan(x-a)\right]dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log\left|\cos(x-b)\right|+\log\left|\cos(x-a)\right|\right]$$

$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+\text{C}$$

Question 23:

$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx$ is equal to

A. tan x + cot x + C

B. tan x + cosec x + C

C. − tan x + cot x + C

D. tan x + sec x + C

Answer

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx$$
$$= \int \left(\sec^2 x - \operatorname{cosec}^2 x \right) dx$$
$$= \tan x + \cot x + C$$

Hence, the correct Answer is A.

Question 24:

$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ equals

A. − cot (ex^x) + C

B. tan (xe^x) + C

C. tan (e^x) + C

D. cot (e^x) + C

Answer

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$$

Let ex^x = t

$$\Rightarrow \left(e^{x}\cdot x + e^{x}\cdot 1\right)dx = dt$$

$$e^{x}\left(x+1\right)dx = dt$$

$$\therefore \int \frac{e^{x}\left(1+x\right)}{\cos^{2}\left(e^{x}x\right)}\,dx = \int \frac{dt}{\cos^{2} t}$$

$$= \int \sec^{2} t\ dt$$

$$= \tan t + C$$

$$= \tan\left(e^{x}\cdot x\right) + C$$

Hence, the correct Answer is B.

Exercise 7.4

Question 1:

$$\frac{3x^2}{x^6+1}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\, dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$
$$= \tan^{1} t + C$$
$$= \tan^{-1}\left(x^3\right) + C$$

Question 2:

$$\frac{1}{\sqrt{1+4x^2}}$$

Answer

Let $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right] + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}} dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2+1}}$$

Answer

Let $2 - x = t$

$\Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}} dx = -\int \frac{1}{\sqrt{t^2+1}} dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right| + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}} dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right| + C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right| + C$$

Question 4:

$$\frac{1}{\sqrt{9-25x^2}}$$

Answer

Let $5x = t$

∴ 5dx = dt

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}\,dx = \frac{1}{5}\int \frac{1}{9-t^2}\,dt$$

$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}\,dt$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Question 5:

$$\frac{3x}{1+2x^4}$$

Answer

Let $\sqrt{2}x^2 = t$

$\therefore\ 2\sqrt{2}x\ dx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Question 6:

$$\frac{x^2}{1-x^6}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\, dx = dt$

$$\Rightarrow \int \frac{x^2}{1-x^6} dx = \frac{1}{3} \int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

Answer

$$\int \frac{x-1}{\sqrt{x^2-1}}\, dx = \int \frac{x}{\sqrt{x^2-1}}\, dx - \int \frac{1}{\sqrt{x^2-1}}\, dx \qquad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}} dx$, let $x^2 - 1 = t \;\Rightarrow\; 2x\; dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}} dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\int t^{-\frac{1}{2}} dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Answer

Let $x^3 = t$

$\Rightarrow 3x^2\,dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$$

Answer

Let $\tan x = t$

$\therefore \sec^2 x\,dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$
$$= \log\left|t + \sqrt{t^2 + 4}\right| + C$$
$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Answer

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x + 1 = t$

$\therefore\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{t^2 + 1}} dt$$
$$= \log\left|t + \sqrt{t^2 + 1}\right| + C$$
$$= \log\left|(x+1) + \sqrt{(x+1)^2 + 1}\right| + C$$
$$= \log\left|(x+1) + \sqrt{x^2 + 2x + 2}\right| + C$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Answer

$$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{(3x+1)^2+(2)^2}dx$$

Let $(3x+1)=t$

$\therefore\ 3dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int \frac{1}{t^2+2^2}dt$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Answer

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$.

Therefore,

$7-\left(x^2+6x+9-9\right)$

$=16-\left(x^2+6x+9\right)$

$=16-(x+3)^2$

$=(4)^2-(x+3)^2$

$$\therefore\ \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let $x+3=t$

$\Rightarrow\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Answer

$(x-1)(x-2)$ can be written as x^2-3x+2.

Therefore,

x^2-3x+2

$=x^2-3x+\frac{9}{4}-\frac{9}{4}+2$

$=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$

$=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$

$$\therefore \int\frac{1}{\sqrt{(x-1)(x-2)}}dx=\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

Question 14:

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Answer

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$.

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx = \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Question 15:

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Answer

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$$x^2-(a+b)x+ab$$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\Rightarrow \int \frac{1}{\sqrt{(x-a)(x-b)}}dx=\int \frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$$

Let $x-\left(\frac{a+b}{2}\right)=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx=\int \frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+\text{C}$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+\text{C}$$

Question 16:

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Answer

Let $4x+1=A\frac{d}{dx}\left(2x^2+x-3\right)+B$

$\Rightarrow 4x+1=A(4x+1)+B$

$\Rightarrow 4x+1=4Ax+A+B$

Equating the coefficients of x and constant term on both sides, we obtain

4A = 4 ⇒ A = 1

A + B = 1 ⇒ B = 0

Let $2x^2 + x - 3 = t$
∴ $(4x + 1)\, dx = dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{1}{\sqrt{t}}dt$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{2x^2+x-3} + C$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Answer

$$\text{Let } x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad ...(1)$$
$$\Rightarrow x+2 = A(2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$B = 2$$

From (1), we obtain

$(x+2)=\frac{1}{2}(2x)+2$

Then, $\int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$

In $\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx$, let $x^2-1=t \Rightarrow 2xdx = dt$

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\left[2\sqrt{t}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

Then, $\int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$

From equation (2), we obtain

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+\mathrm{C}$$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

Answer

Let $5x-2 = A\frac{d}{dx}\left(1+2x+3x^2\right)+B$

$\Rightarrow 5x-2 = A\left(2+6x\right)+B$

Equating the coefficient of x and constant term on both sides, we obtain

$5 = 6A \Rightarrow A = \frac{5}{6}$

$2A + B = -2 \Rightarrow B = -\frac{11}{3}$

$\therefore 5x - 2 = \frac{5}{6}(2+6x) + \left(-\frac{11}{3}\right)$

$$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}dx$$

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$

Let $I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$ and $I_2 = \int \frac{1}{1+2x+3x^2}dx$

$$\therefore \int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \qquad ...(1)$$

$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$

Let $1 + 2x + 3x^2 = t$

$\Rightarrow (2+6x)dx = dt$

$\therefore I_1 = \int \frac{dt}{t}$

$I_1 = \log|t|$

$$I_1 = \log\left|1+2x+3x^2\right| \qquad ...(2)$$

$I_2 = \int \frac{1}{1+2x+3x^2}dx$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right]-\frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]+C$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

Question 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Answer

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}=\frac{6x+7}{\sqrt{x^2-9x+20}}$$

Let $6x+7=A\frac{d}{dx}\left(x^2-9x+20\right)+B$

$\Rightarrow 6x+7=A(2x-9)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \qquad ...(1)$$

Then,

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2 - 9x + 20 = t$

$\Rightarrow (2x-9)\,dx = dt$

$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$

$I_1 = 2\sqrt{t}$

$$I_1 = 2\sqrt{x^2-9x+20} \qquad ...(2)$$

and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$
$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$
$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$
$$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$
$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$
$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

Question 20:

$$\frac{x+2}{\sqrt{4x-x^2}}$$

Answer

Let $x+2=A\frac{d}{dx}\left(4x-x^2\right)+B$

$\Rightarrow x+2=A(4-2x)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$

$$4A + B = 2 \Rightarrow B = 4$$

$$\Rightarrow (x+2) = -\frac{1}{2}(4-2x)+4$$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x - x^2 = t$

$\Rightarrow (4-2x)\,dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad ...(2)$$

$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\Rightarrow 4x - x^2 = -(-4x + x^2)$$

$$= (-4x + x^2 + 4 - 4)$$

$$= 4 - (x-2)^2$$

$$= (2)^2 - (x-2)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2-(x-2)^2}}\,dx = \sin^{-1}\left(\frac{x-2}{2}\right) \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\,dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad \ldots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+\left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \qquad \ldots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right]+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$

$$= \sqrt{x^2+2x+3}+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$

Question 22:

$$\frac{x+3}{x^2-2x-5}$$

Answer

$$\text{Let } (x+3) = A\frac{d}{dx}\left(x^2-2x-5\right)+B$$

$$(x+3) = A(2x-2)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A+B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2)+4$$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

Let $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$ and $I_2 = \int \frac{1}{x^2-2x-5}dx$

$$\therefore \int \frac{x+3}{\left(x^2-2x-5\right)}dx = \frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$

Let $x^2-2x-5=t$

$\Rightarrow (2x-2)\,dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log\left|x^2-2x-5\right| \qquad ...(2)$$

$$\begin{aligned} I_2 &= \int \frac{1}{x^2-2x-5}dx \\ &= \int \frac{1}{\left(x^2-2x+1\right)-6}dx \\ &= \int \frac{1}{(x-1)^2+\left(\sqrt{6}\right)^2}dx \\ &= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \qquad ...(3) \end{aligned}$$

Substituting (2) and (3) in (1), we obtain

$$\begin{aligned} \int \frac{x+3}{x^2-2x-5}dx &= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C \\ &= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C \end{aligned}$$

Question 23:

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Answer

Let $5x+3=A\frac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3=A\left(2x+4\right)+B$

Equating the coefficients of x and constant term, we obtain

$2A=5\Rightarrow A=\frac{5}{2}$

$4A+B=3\Rightarrow B=-7$

$\therefore 5x+3=\frac{5}{2}\left(2x+4\right)-7$

$$\Rightarrow \int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int\frac{\frac{5}{2}\left(2x+4\right)-7}{\sqrt{x^2+4x+10}}dx$$

$$=\frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx-7\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1=\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2=\int\frac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\frac{5}{2}I_1-7I_2 \qquad ...(1)$$

Then, $I_1=\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10=t$

$\therefore \left(2x+4\right)dx=dt$

$$\Rightarrow I_1=\int\frac{dt}{t}=2\sqrt{t}=2\sqrt{x^2+4x+10} \qquad ...(2)$$

$$I_2=\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

$$=\int\frac{1}{\sqrt{\left(x^2+4x+4\right)+6}}dx$$

$$=\int\frac{1}{\left(x+2\right)^2+\left(\sqrt{6}\right)^2}dx$$

$$=\log\left|\left(x+2\right)\sqrt{x^2+4x+10}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$
$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Question 24:

$\int \frac{dx}{x^2+2x+2}$ equals

A. x $\tan^{-1}$ (x + 1) + C

B. $\tan^{-1}$ (x + 1) + C

C. (x + 1) $\tan^{-1}$ x + C

D. $\tan^{-1}$ x + C

Answer

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$
$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$
$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the correct Answer is B.

Question 25:

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

A. $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

B. $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

C. $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

D. $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Answer

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$= \int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$= \int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}} = \sin^{-1}\frac{y}{a}+C\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct Answer is B.

Exercise 7.5

Question 1:

$$\frac{x}{(x+1)(x+2)}$$

Answer

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$$\Rightarrow x = A(x+2) + B(x+1)$$

Equating the coefficients of x and constant term, we obtain

A + B = 1

2A + B = 0

On solving, we obtain

A = −1 and B = 2

$$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$$

$$\Rightarrow \int \frac{x}{(x+1)(x+2)} dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= \log(x+2)^2 - \log|x+1| + C$$

$$= \log\frac{(x+2)^2}{(x+1)} + C$$

Question 2:

$$\frac{1}{x^2-9}$$

Answer

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$$1 = A(x-3) + B(x+3)$$

Equating the coefficients of x and constant term, we obtain

A + B = 0

−3A + 3B = 1

On solving, we obtain

$A=-\frac{1}{6}$ and $B=\frac{1}{6}$

$$\therefore \frac{1}{(x+3)(x-3)}=\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}$$

$$\Rightarrow \int\frac{1}{(x^2-9)}dx=\int\left(\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}\right)dx$$

$$=-\frac{1}{6}\log|x+3|+\frac{1}{6}\log|x-3|+C$$

$$=\frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right|+C$$

Question 3:

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Answer

Let $\frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$$3x-1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \qquad ...(1)$$

Substituting x = 1, 2, and 3 respectively in equation (1), we obtain

A = 1, B = −5, and C = 4

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}$$

$$\Rightarrow \int\frac{3x-1}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}\right\}dx$$

$$=\log|x-1|-5\log|x-2|+4\log|x-3|+C$$

Question 4:

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Answer

Let $\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$$x=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \qquad ...(1)$$

Substituting x = 1, 2, and 3 respectively in equation (1), we obtain

$$A=\frac{1}{2},\ B=-2,\ \text{and } C=\frac{3}{2}$$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}$$

$$\Rightarrow \int\frac{x}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}\right\}dx$$

$$=\frac{1}{2}\log|x-1|-2\log|x-2|+\frac{3}{2}\log|x-3|+C$$

Question 5:

$$\frac{2x}{x^2+3x+2}$$

Answer

Let $\frac{2x}{x^2+3x+2}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$$2x=A(x+2)+B(x+1) \qquad ...(1)$$

Substituting x = −1 and −2 in equation (1), we obtain

A = −2 and B = 4

$$\therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$$

$$\Rightarrow \int \frac{2x}{(x+1)(x+2)} dx = \int \left\{ \frac{4}{(x+2)} - \frac{2}{(x+1)} \right\} dx$$

$$= 4 \log|x+2| - 2 \log|x+1| + C$$

Question 6:

$$\frac{1-x^2}{x(1-2x)}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$$

Let $\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$

$$\Rightarrow (2-x) = A(1-2x) + Bx \qquad ...(1)$$

Substituting x = 0 and $\frac{1}{2}$ in equation (1), we obtain

A = 2 and B = 3

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left\{\frac{2}{x}+\frac{3}{(1-2x)}\right\}$$

$$\Rightarrow \int\frac{1-x^2}{x(1-2x)}dx=\int\left\{\frac{1}{2}+\frac{1}{2}\left(\frac{2}{x}+\frac{3}{1-2x}\right)\right\}dx$$

$$=\frac{x}{2}+\log|x|+\frac{3}{2(-2)}\log|1-2x|+C$$

$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$$

Question 7:

$$\frac{x}{(x^2+1)(x-1)}$$

Answer

Let $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-1)}$

$$x=(Ax+B)(x-1)+C(x^2+1)$$

$$x=Ax^2-Ax+Bx-B+Cx^2+C$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + C = 0

−A + B = 1

−B + C = 0

On solving these equations, we obtain

$$A=-\frac{1}{2}, B=\frac{1}{2}, \text{ and } C=\frac{1}{2}$$

From equation (1), we obtain

$$\therefore \frac{x}{(x^2+1)(x-1)} = \frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{(x-1)}$$

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

Consider $\int \frac{2x}{x^2+1}dx$, let $(x^2+1) = t \Rightarrow 2x\,dx = dt$

$$\Rightarrow \int \frac{2x}{x^2+1}dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\therefore \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

Question 8:

$$\frac{x}{(x-1)^2(x+2)}$$

Answer

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Substituting x = 1, we obtain

$$B = \frac{1}{3}$$

Equating the coefficients of x^2 and constant term, we obtain

A + C = 0

−2A + 2B + C = 0 On

solving, we obtain

$A=\frac{2}{9}$ and $C=\frac{-2}{9}$

$$\therefore \frac{x}{(x-1)^2(x+2)}=\frac{2}{9(x-1)}+\frac{1}{3(x-1)^2}-\frac{2}{9(x+2)}$$

$$\Rightarrow \int\frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\int\frac{1}{(x-1)}dx+\frac{1}{3}\int\frac{1}{(x-1)^2}dx-\frac{2}{9}\int\frac{1}{(x+2)}dx$$

$$=\frac{2}{9}\log|x-1|+\frac{1}{3}\left(\frac{-1}{x-1}\right)-\frac{2}{9}\log|x+2|+\mathrm{C}$$

$$=\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+\mathrm{C}$$

Question 9:

$$\frac{3x+5}{x^3-x^2-x+1}$$

Answer

$$\frac{3x+5}{x^3-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$$

Let $\frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$

$$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$$
$$3x+5=A(x^2-1)+B(x+1)+C(x^2+1-2x) \qquad ...(1)$$

Substituting x = 1 in equation (1), we obtain

B = 4

Equating the coefficients of x^2 and x, we obtain

A + C = 0

B − 2C = 3

On solving, we obtain

$A=-\frac{1}{2}$ and $C=\frac{1}{2}$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)} dx = -\frac{1}{2}\int \frac{1}{x-1} dx + 4\int \frac{1}{(x-1)^2} dx + \frac{1}{2}\int \frac{1}{(x+1)} dx$$

$$= -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + \mathrm{C}$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + \mathrm{C}$$

Question 10:

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Answer

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$

Let $\dfrac{2x-3}{(x+1)(x-1)(2x+3)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)} + \dfrac{C}{(2x+3)}$

$$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

$$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$$

$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of x^2 and x, we obtain

$$B = -\frac{1}{10},\ A = \frac{5}{2},\text{ and } C = -\frac{24}{5}$$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)}dx = \frac{5}{2}\int \frac{1}{(x+1)}dx - \frac{1}{10}\int \frac{1}{x-1}dx - \frac{24}{5}\int \frac{1}{(2x+3)}dx$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

Question 11:

$$\frac{5x}{(x+1)(x^2-4)}$$

Answer

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \qquad \text{...(1)}$$

Substituting x = −1, −2, and 2 respectively in equation (1), we obtain

$$A = \frac{5}{3},\ B = -\frac{5}{2}, \text{ and } C = \frac{5}{6}$$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)}dx = \frac{5}{3}\int \frac{1}{(x+1)}dx - \frac{5}{2}\int \frac{1}{(x+2)}dx + \frac{5}{6}\int \frac{1}{(x-2)}dx$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

Question 12:

$$\frac{x^3+x+1}{x^2-1}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing ($x^3 + x + 1$) by $x^2 - 1$, we obtain

$$\frac{x^3+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}$$

Let $\frac{2x+1}{x^2-1}=\frac{A}{(x+1)}+\frac{B}{(x-1)}$

$$2x+1=A(x-1)+B(x+1) \qquad ...(1)$$

Substituting x = 1 and −1 in equation (1), we obtain

$$A=\frac{1}{2} \text{ and } B=\frac{3}{2}$$

$$\therefore \frac{x^3+x+1}{x^2-1}=x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$$

$$\Rightarrow \int\frac{x^3+x+1}{x^2-1}dx=\int x\,dx+\frac{1}{2}\int\frac{1}{(x+1)}dx+\frac{3}{2}\int\frac{1}{(x-1)}dx$$

$$=\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1|+C$$

Question 13:

$$\frac{2}{(1-x)(1+x^2)}$$

Answer

Let $\frac{2}{(1-x)(1+x^2)}=\frac{A}{(1-x)}+\frac{Bx+C}{(1+x^2)}$

$$2=A(1+x^2)+(Bx+C)(1-x)$$

$$2=A+Ax^2+Bx-Bx^2+C-Cx$$

Equating the coefficient of x^2, x, and constant term, we obtain

A − B = 0

B − C = 0

A + C = 2

On solving these equations, we obtain

A = 1, B = 1, and C = 1

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|x-1| + \frac{1}{2}\log\left|1+x^2\right| + \tan^{-1} x + C$$

Question 14:

$$\frac{3x-1}{(x+2)^2}$$

Answer

$$\text{Let } \frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$\Rightarrow 3x-1 = A(x+2) + B$$

Equating the coefficient of x and constant term, we obtain

A = 3

2A + B = −1 ⇒ B = −7

$$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

$$\Rightarrow \int \frac{3x-1}{(x+2)^2}\,dx = 3\int \frac{1}{(x+2)}dx - 7\int \frac{x}{(x+2)^2}\,dx$$

$$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + C$$

Question 15:

$$\frac{1}{x^4-1}$$

Answer

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1)$$

$$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

Equating the coefficient of x^3, x^2, x, and constant term, we obtain

$$A+B+C=0$$

$$-A+B+D=0$$

$$A+B-C=0$$

$$-A+B-D=1$$

On solving these equations, we obtain

$$A = -\frac{1}{4}, B = \frac{1}{4}, C = 0, \text{ and } D = -\frac{1}{2}$$

$$\therefore \frac{1}{x^4-1} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}$$

$$\Rightarrow \int \frac{1}{x^4-1}dx = -\frac{1}{4}\log|x-1| + \frac{1}{4}\log|x-1| - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C$$

Question 16:

$\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Answer

$$\frac{1}{x(x^n+1)}$$

Multiplying numerator and denominator by x^{n-1}, we obtain

$$\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x^{n-1}x(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}$$

Let $x^n = t \Rightarrow x^{n-1}dx = dt$

$$\therefore \int \frac{1}{x(x^n+1)}dx = \int \frac{x^{n-1}}{x^n(x^n+1)}dx = \frac{1}{n}\int \frac{1}{t(t+1)}dt$$

Let $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$

$$1 = A(1+t) + Bt \qquad ...(1)$$

Substituting t = 0, −1 in equation (1), we obtain

A = 1 and B = −1

$$\therefore \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x\left(x^n+1\right)} dx = \frac{1}{n} \int \left\{ \frac{1}{t} - \frac{1}{(t+1)} \right\} dx$$
$$= \frac{1}{n}\left[\log|t| - \log|t+1|\right] + C$$
$$= -\frac{1}{n}\left[\log\left|x^n\right| - \log\left|x^n+1\right|\right] + C$$
$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$$

Question 17:

$\frac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint: Put sin x = t]

Answer

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{dt}{(1-t)(2-t)}$$

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$

$$1 = A(2-t) + B(1-t) \qquad \ldots(1)$$

Substituting t = 2 and then t = 1 in equation (1), we obtain

A = 1 and B = −1

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int\left\{\frac{1}{1-t}-\frac{1}{(2-t)}\right\}dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

Question 18:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

Answer

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

$$\text{Let } \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$$

$$4x^2+10 = (Ax+B)(x^2+4)+(Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3+4Ax+Bx^2+4B+Cx^3+3Cx+Dx^2+3D$$

$$4x^2+10 = (A+C)x^3+(B+D)x^2+(4A+3C)x+(4B+3D)$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

A + C = 0

B + D = 4

4A + 3C = 0

4B + 3D = 10

On solving these equations, we obtain

A = 0, B = −2, C = 0, and D = 6

$$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = \int \left\{1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)}\right\} dx$$

$$= \int \left\{1 + \frac{2}{x^2+(\sqrt{3})^2} - \frac{6}{x^2+2^2}\right\}$$

$$= x + 2\left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}}\right) - 6\left(\frac{1}{2} \tan^{-1} \frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \frac{x}{2} + C$$

Question 19:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Answer

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Let $x^2 = t \Rightarrow 2x\, dx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dt}{(t+1)(t+3)} \qquad \ldots(1)$$

$$\text{Let } \frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$$

$$1 = A(t+3) + B(t+1) \qquad \ldots(1)$$

Substituting t = −3 and t = −1 in equation (1), we obtain

$A=\frac{1}{2}$ and $B=-\frac{1}{2}$

$$\therefore \frac{1}{(t+1)(t+3)}=\frac{1}{2(t+1)}-\frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)}dx=\int\left\{\frac{1}{2(t+1)}-\frac{1}{2(t+3)}\right\}dt$$

$$=\frac{1}{2}\log|(t+1)|-\frac{1}{2}\log|t+3|+C$$

$$=\frac{1}{2}\log\left|\frac{t+1}{t+3}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$$

Question 20:

$$\frac{1}{x(x^4-1)}$$

Answer

$$\frac{1}{x(x^4-1)}$$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x(x^4-1)}=\frac{x^3}{x^4(x^4-1)}$$

$$\therefore \int \frac{1}{x(x^4-1)}dx=\int \frac{x^3}{x^4(x^4-1)}dx$$

Let $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\therefore \int \frac{1}{x(x^4-1)}dx=\frac{1}{4}\int \frac{dt}{t(t-1)}$$

Let $\dfrac{1}{t(t-1)}=\dfrac{A}{t}+\dfrac{B}{(t-1)}$

$1=A(t-1)+Bt \qquad ...(1)$

Substituting t = 0 and 1 in (1), we obtain

A = −1 and B = 1

$\Rightarrow \dfrac{1}{t(t+1)}=\dfrac{-1}{t}+\dfrac{1}{t-1}$

$$\Rightarrow \int \frac{1}{x(x^4-1)}dx=\frac{1}{4}\int\left\{\frac{-1}{t}+\frac{1}{t-1}\right\}dt$$

$$=\frac{1}{4}\left[-\log|t|+\log|t-1|\right]+C$$

$$=\frac{1}{4}\log\left|\frac{t-1}{t}\right|+C$$

$$=\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$$

Question 21:

$\dfrac{1}{(e^x-1)}$ [Hint: Put $e^x = t$]

Answer

$\dfrac{1}{(e^x-1)}$

Let $e^x = t \Rightarrow e^x\,dx = dt$

$$\Rightarrow \int \frac{1}{e^x-1}dx=\int\frac{1}{t-1}\times\frac{dt}{t}=\int\frac{1}{t(t-1)}dt$$

Let $\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{t-1}$

$1=A(t-1)+Bt \quad ...(1)$

Substituting t = 1 and t = 0 in equation (1), we obtain

A = −1 and B = 1

$\therefore \frac{1}{t(t-1)}=\frac{-1}{t}+\frac{1}{t-1}$

$$\Rightarrow \int\frac{1}{t(t-1)}dt=\log\left|\frac{t-1}{t}\right|+C$$

$$=\log\left|\frac{e^x-1}{e^x}\right|+C$$

Question 22:

$\int\frac{xdx}{(x-1)(x-2)}$ equals

A. $\log\left|\frac{(x-1)^2}{x-2}\right|+C$

B. $\log\left|\frac{(x-2)^2}{x-1}\right|+C$

C. $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|+C$

D. $\log\left|(x-1)(x-2)\right|+C$

Answer

Let $\frac{x}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

$x=A(x-2)+B(x-1) \quad ...(1)$

Substituting x = 1 and 2 in (1), we obtain

A = −1 and B = 2

$$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Hence, the correct Answer is B.

Question 23:

$\int \frac{dx}{x(x^2+1)}$ equals

A. $\log|x| - \frac{1}{2}\log(x^2+1) + C$

B. $\log|x| + \frac{1}{2}\log(x^2+1) + C$

C. $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

D. $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Answer

Let $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$1 = A(x^2+1) + (Bx+C)x$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

C = 0

A = 1

On solving these equations, we obtain

A = 1, B = −1, and C = 0

$$\therefore \frac{1}{x\left(x^2+1\right)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x\left(x^2+1\right)} dx = \int \left\{ \frac{1}{x} - \frac{x}{x^2+1} \right\} dx$$

$$= \log|x| - \frac{1}{2}\log\left|x^2+1\right| + C$$

Hence, the correct Answer is A.

Exercise 7.6

Question 1:

x sin x

Answer

Let I = $\int x \sin x \, dx$

Taking x as first function and sin x as second function and integrating by parts, we obtain

$$I = x\int \sin x \; dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin x \; dx\right\}dx$$
$$= x(-\cos x) - \int 1\cdot(-\cos x)\,dx$$
$$= -x\cos x + \sin x + C$$

Question 2:

$x \sin 3x$

Answer

Let I = $\int x \sin 3x \, dx$

Taking x as first function and sin 3x as second function and integrating by parts, we obtain

$$I = x\int \sin 3x \; dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin 3x \; dx\right\}$$
$$= x\left(\frac{-\cos 3x}{3}\right) - \int 1\cdot\left(\frac{-\cos 3x}{3}\right)dx$$
$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x \; dx$$
$$= \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

Question 3:

$x^2 e^x$

Answer

Let $I=\int x^2e^x dx$

Taking x^2 as first function and e^x as second function and integrating by parts, we obtain

$$I=x^2\int e^x\,dx-\int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx$$
$$=x^2e^x-\int 2x\cdot e^x dx$$
$$=x^2e^x-2\int x\cdot e^x dx$$

Again integrating by parts, we obtain

$$=x^2e^x-2\left[x\cdot\int e^x dx-\int\left\{\left(\frac{d}{dx}x\right)\cdot\int e^x dx\right\}dx\right]$$
$$=x^2e^x-2\left[xe^x-\int e^x dx\right]$$
$$=x^2e^x-2\left[xe^x-e^x\right]$$
$$=x^2e^x-2xe^x+2e^x+\mathrm{C}$$
$$=e^x\left(x^2-2x+2\right)+\mathrm{C}$$

Question 4:

x logx

Answer

Let $I=\int x\log x dx$

Taking log x as first function and x as second function and integrating by parts, we obtain

$$I=\log x\int x\;dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x\;dx\right\}dx$$
$$=\log x\cdot\frac{x^2}{2}-\int\frac{1}{x}\cdot\frac{x^2}{2}dx$$
$$=\frac{x^2\log x}{2}-\int\frac{x}{2}dx$$
$$=\frac{x^2\log x}{2}-\frac{x^2}{4}+\mathrm{C}$$

Question 5:

x log 2x

Answer

Let $I = \int x \log 2x dx$

Taking log 2x as first function and x as second function and integrating by parts, we obtain

$$I = \log 2x \int x\ dx - \int \left\{ \left(\frac{d}{dx} 2 \log x \right) \int x\ dx \right\} dx$$

$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + \text{C}$$

Question 6:

x^2 log x

Answer

Let $I = \int x^2 \log x\ dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I = \log x \int x^2\ dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx$$

$$= \log x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + \text{C}$$

Question 7:

$x\sin^{-1}x$

Answer

Let $I=\int x\sin^{-1}x\,dx$

Taking $\sin^{-1}x$ as first function and x as second function and integrating by parts, we obtain

$$I=\sin^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int x\,dx\right\}dx$$

$$=\sin^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\int\sqrt{1-x^2}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right\}$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x-\sin^{-1}x\right\}+\mathrm{C}$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{x}{4}\sqrt{1-x^2}+\frac{1}{4}\sin^{-1}x-\frac{1}{2}\sin^{-1}x+\mathrm{C}$$

$$=\frac{1}{4}\left(2x^2-1\right)\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+\mathrm{C}$$

Question 8:

$x\tan^{-1}x$

Answer

Let $I=\int x\tan^{-1}x\,dx$

Taking $\tan^{-1} x$ as first function and x as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int x\,dx - \int\left\{\left(\frac{d}{dx}\tan^{-1} x\right)\int x\,dx\right\}dx$$

$$= \tan^{-1} x\left(\frac{x^2}{2}\right) - \int\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int\left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$

Question 9:

$x\cos^{-1} x$

Answer

Let $I = \int x\cos^{-1} x\,dx$

Taking $\cos^{-1}$ x as first function and x as second function and integrating by parts, we obtain

$$I=\cos^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int x\,dx\right\}dx$$

$$=\cos^{-1}x\frac{x^2}{2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\sqrt{1-x^2}+\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\sqrt{1-x^2}dx-\frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}I_1-\frac{1}{2}\cos^{-1}x \qquad \ldots(1)$$

where, $I_1=\int\sqrt{1-x^2}dx$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{d}{dx}\sqrt{1-x^2}\int x\,dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-2x}{2\sqrt{1-x^2}}.x\,dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{\int\sqrt{1-x^2}\,dx+\int\frac{-dx}{\sqrt{1-x^2}}\right\}$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{I_1+\cos^{-1}x\right\}$$

$$\Rightarrow 2I_1=x\sqrt{1-x^2}-\cos^{-1}x$$

$$\therefore I_1=\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x$$

Substituting in (1), we obtain

$$I=\frac{x\cos^{-1}x}{2}-\frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x\right)-\frac{1}{2}\cos^{-1}x$$

$$=\frac{\left(2x^2-1\right)}{4}\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C$$

Question 10:

$\left(\sin^{-1} x\right)^2$

Answer

Let $I = \int \left(\sin^{-1} x\right)^2 \cdot 1\, dx$

Taking $\left(\sin^{-1} x\right)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \left(\sin^{-1} x\right) \int 1\, dx - \int \left\{ \frac{d}{dx}\left(\sin^{-1} x\right)^2 \cdot \int 1 \cdot dx \right\} dx$$

$$= \left(\sin^{-1} x\right)^2 \cdot x - \int \frac{2\sin^{-1} x}{\sqrt{1-x^2}} \cdot x\, dx$$

$$= x\left(\sin^{-1} x\right)^2 + \int \sin^{-1} x \cdot \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx$$

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \int \frac{-2x}{\sqrt{1-x^2}}\, dx - \int \left\{\left(\frac{d}{dx}\sin^{-1} x\right) \int \frac{-2x}{\sqrt{1-x^2}} dx\right\} dx\right]$$

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}\, dx\right]$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\, \sin^{-1} x - \int 2\, dx$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\, \sin^{-1} x - 2x + \mathrm{C}$$

Question 11:

$\frac{x\cos^{-1} x}{\sqrt{1-x^2}}$

Answer

Let $I = \int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}\, dx$

$$I=\frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^2}}\cdot\cos^{-1}x\,dx$$

Taking $\cos^{-1}x$ as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$$I=\frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$

$$=\frac{-1}{2}\left[\cos^{-1}x\cdot2\sqrt{1-x^2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot2\sqrt{1-x^2}\,dx\right]$$

$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+\int2\,dx\right]$$

$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+2x\right]+C$$

$$=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$$

Question 12:

$x\sec^2x$

Answer

Let $I=\int x\sec^2x\,dx$

Taking x as first function and $\sec^2x$ as second function and integrating by parts, we obtain

$$I=x\int\sec^2x\,dx-\int\left\{\left\{\frac{d}{dx}x\right\}\int\sec^2x\,dx\right\}dx$$

$$=x\tan x-\int1\cdot\tan x\,dx$$

$$=x\tan x+\log\left|\cos x\right|+C$$

Question 13:

$\tan^{-1}x$

Answer

Let $I = \int 1 \cdot \tan^{-1} x dx$

Taking $\tan^{-1} x$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int 1 dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int 1 \cdot dx \right\} dx$$

$$= \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \tan^{-1} x - \frac{1}{2} \log\left|1+x^2\right| + \mathrm{C}$$

$$= x \tan^{-1} x - \frac{1}{2} \log\left(1+x^2\right) + \mathrm{C}$$

Question 14:

$x(\log x)^2$

Answer

$$I = \int x(\log x)^2 \, dx$$

Taking $(\log x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = (\log x)^2 \int x \, dx - \int \left[\left\{ \left(\frac{d}{dx} \log x \right)^2 \right\} \int x \, dx \right] dx$$

$$= \frac{x^2}{2} (\log x)^2 - \left[\int 2 \log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \int x \log x \, dx$$

Again integrating by parts, we obtain

$$I=\frac{x^2}{2}(\log x)^2-\left[\log x\int x\,dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x\,dx\right\}dx\right]$$
$$=\frac{x^2}{2}(\log x)^2-\left[\frac{x^2}{2}-\log x-\int\frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$
$$=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{1}{2}\int x\,dx$$
$$=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{x^2}{4}+\text{C}$$

Question 15:

$\left(x^2+1\right)\log x$

Answer

Let $I=\int\left(x^2+1\right)\log x\,dx=\int x^2\log x\,dx+\int\log x\,dx$

Let I = I_1 + I_2 … (1)

Where, $I_1=\int x^2\log x\,dx$ and $I_2=\int\log x\,dx$

$I_1=\int x^2\log x dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I_1=\log x-\int x^2dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x^2dx\right\}dx$$
$$=\log x\cdot\frac{x^3}{3}-\int\frac{1}{x}\cdot\frac{x^3}{3}dx$$
$$=\frac{x^3}{3}\log x-\frac{1}{3}\left(\int x^2dx\right)$$
$$=\frac{x^3}{3}\log x-\frac{x^3}{9}+\text{C}_1\qquad\ldots(2)$$

$I_2=\int\log x\,dx$

Taking log x as first function and 1 as second function and integrating by parts, we obtain

$$I_2 = \log x \int 1 \cdot dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int 1 \cdot dx \right\}$$
$$= \log x \cdot x - \int \frac{1}{x} \cdot x dx$$
$$= x \log x - \int 1 dx$$
$$= x \log x - x + C_2 \quad \dots (3)$$

Using equations (2) and (3) in (1), we obtain

$$I = \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 + x \log x - x + C_2$$
$$= \frac{x^3}{3} \log x - \frac{x^3}{9} + x \log x - x + (C_1 + C_2)$$
$$= \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C$$

Question 16:

$e^x (\sin x + \cos x)$

Answer

Let $I = \int e^x (\sin x + \cos x) dx$

Let $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$\Rightarrow I = \int e^x \{ f(x) + f'(x) \} dx$

It is known that, $\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

Question 17:

$$\frac{xe^x}{(1+x)^2}$$

Answer

Let $I = \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx$

$= \int e^x \left\{ \frac{1+x-1}{(1+x)^2} \right\} dx$

$= \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$

Let $f(x) = \frac{1}{1+x}$ $\Rightarrow$ $f'(x) = \frac{-1}{(1+x)^2}$

$\Rightarrow \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{ f(x) + f'(x) \right\} dx$

It is known that, $\int e^x \left\{ f(x) + f'(x) \right\} dx = e^x f(x) + \mathrm{C}$

$\therefore \int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + \mathrm{C}$

Question 18:

$e^x \left(\frac{1+\sin x}{1+\cos x} \right)$

Answer

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$

$$=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2}e^x\cdot\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$

$$=\frac{1}{2}e^2\left(1+\tan\frac{x}{2}\right)^2$$

$$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$\frac{e^x(1+\sin x)dx}{(1+\cos x)}=e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right] \qquad ...(1)$$

Let $\tan\frac{x}{2}=f(x)$ $\Rightarrow$ $f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

From equation (1), we obtain

$$\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx=e^x\tan\frac{x}{2}+C$$

Question 19:

$e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Answer

$$\text{Let } I=\int e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]dx$$

Also, let $\frac{1}{x}=f(x)$ ⇒ $f'(x)=\frac{-1}{x^2}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+\text{C}$

$$\therefore I=\frac{e^x}{x}+\text{C}$$

Question 20:

$\frac{(x-3)e^x}{(x-1)^3}$

Answer

$$\int e^x\left\{\frac{x-3}{(x-1)^3}\right\}dx=\int e^x\left\{\frac{x-1-2}{(x-1)^3}\right\}dx$$

$$=\int e^x\left\{\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right\}dx$$

Let $f(x)=\frac{1}{(x-1)^2}$ ⇒ $f'(x)=\frac{-2}{(x-1)^3}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+\text{C}$

$$\therefore \int e^x\left\{\frac{(x-3)}{(x-1)^2}\right\}dx=\frac{e^x}{(x-1)^2}+\text{C}$$

Question 21:

$e^{2x}\sin x$

Answer

Let $I = \int e^{2x} \sin x \, dx$ ...(1)

Integrating by parts, we obtain

$$I = \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\cos x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \qquad \text{[From (1)]}$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \cdot \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow I = \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5} [2 \sin x - \cos x] + C$$

Question 22:

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

Answer

Let $x = \tan \theta \Rightarrow dx = \sec^2 \theta \, d\theta$

$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)$ = 2θ

⇒ $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx=\int 2\theta\cdot\sec^2\theta\,d\theta = 2\int\theta\cdot\sec^2\theta\,d\theta$

Integrating by parts, we obtain

$$2\left[\theta\cdot\int\sec^2\theta d\theta-\int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]$$
$$=2\left[\theta\cdot\tan\theta-\int\tan\theta d\theta\right]$$
$$=2\left[\theta\tan\theta+\log\left|\cos\theta\right|\right]+C$$
$$=2\left[x\tan^{-1}x+\log\left|\frac{1}{\sqrt{1+x^2}}\right|\right]+C$$
$$=2x\tan^{-1}x+2\log\left(1+x^2\right)^{-\frac{1}{2}}+C$$
$$=2x\tan^{-1}x+2\left[-\frac{1}{2}\log\left(1+x^2\right)\right]+C$$
$$=2x\tan^{-1}x-\log\left(1+x^2\right)+C$$

Question 23:

$\int x^2e^{x^3}dx$ equals

(A) $\frac{1}{3}e^{x^3}+C$ (B) $\frac{1}{3}e^{x^2}+C$

(C) $\frac{1}{2}e^{x^3}+C$ (D) $\frac{1}{3}e^{x^2}+C$

Answer

Let $I=\int x^2e^{x^3}dx$

Also, let $x^3=t \Rightarrow 3x^2dx=dt$

$$\Rightarrow I = \frac{1}{3}\int e^t\,dt$$
$$= \frac{1}{3}\left(e^t\right) + C$$
$$= \frac{1}{3}e^{x^3} + C$$

Hence, the correct Answer is A.

Question 24:

$\int e^x \sec x\left(1+\tan x\right)dx$ equals

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

Answer

$$\int e^x \sec x\left(1+\tan x\right)dx$$

Let $I = \int e^x \sec x\left(1+\tan x\right)dx = \int e^x\left(\sec x + \sec x \tan x\right)dx$

Also, let $\sec x = f(x)$ $\Rightarrow$ $\sec x \tan x = f'(x)$

It is known that, $\int e^x\left\{f(x)+f'(x)\right\}dx = e^x f(x) + C$

$$\therefore I = e^x \sec x + C$$

Hence, the correct Answer is B.

Exercise 7.7

Question 1:

$\sqrt{4-x^2}$

Answer

Let $I = \int\sqrt{4-x^2}\,dx = \int\sqrt{(2)^2-(x)^2}\,dx$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}\,\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\text{C}$

$$\therefore I = \frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}+\text{C}$$

$$=\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+\text{C}$$

Question 2:

$\sqrt{1-4x^2}$

Answer

Let $I = \int\sqrt{1-4x^2}\,dx = \int\sqrt{(1)^2-(2x)^2}\,dx$

Let $2x = t \Rightarrow 2\,dx = dt$

$$\therefore I = \frac{1}{2}\int\sqrt{(1)^2-(t)^2}\,dt$$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\text{C}$

$$\Rightarrow I = \frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+\text{C}$$

$$=\frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t+\text{C}$$

$$=\frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+\text{C}$$

$$=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+\text{C}$$

Question 3:

$\sqrt{x^2+4x+6}$

Answer

$$\begin{aligned}\text{Let } I &= \int\sqrt{x^2+4x+6}\ dx\\ &= \int\sqrt{x^2+4x+4+2}\ dx\\ &= \int\sqrt{\left(x^2+4x+4\right)+2}\ dx\\ &= \int\sqrt{\left(x+2\right)^2+\left(\sqrt{2}\right)^2}\ dx\end{aligned}$$

It is known that, $\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+\text{C}$

$$\begin{aligned}\therefore I &= \frac{(x+2)}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+\text{C}\\ &= \frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+\text{C}\end{aligned}$$

Question 4:

$\sqrt{x^2+4x+1}$

Answer

$$\begin{aligned}\text{Let } I &= \int\sqrt{x^2+4x+1}\,dx\\ &= \int\sqrt{\left(x^2+4x+4\right)-3}\,dx\\ &= \int\sqrt{\left(x+2\right)^2-\left(\sqrt{3}\right)^2}\,dx\end{aligned}$$

It is known that, $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+\text{C}$

$$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+\text{C}$$

Question 5:

$\sqrt{1-4x-x^2}$

Answer

$$\text{Let } I=\int\sqrt{1-4x-x^2}\,dx$$
$$=\int\sqrt{1-\left(x^2+4x+4-4\right)}\,dx$$
$$=\int\sqrt{1+4-\left(x+2\right)^2}\,dx$$
$$=\int\sqrt{\left(\sqrt{5}\right)^2-\left(x+2\right)^2}\,dx$$

It is known that, $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$$

Question 6:

$\sqrt{x^2+4x-5}$

Answer

$$\text{Let } I=\int\sqrt{x^2+4x-5}\,dx$$
$$=\int\sqrt{\left(x^2+4x+4\right)-9}\,dx$$
$$=\int\sqrt{\left(x+2\right)^2-\left(3\right)^2}\,dx$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$$

Question 7:

$\sqrt{1+3x-x^2}$

Answer

Let $I = \int\sqrt{1+3x-x^2}\,dx$

$$= \int\sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}\,dx$$

$$= \int\sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\,dx$$

$$= \int\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\text{C}$

$$\therefore I = \frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2}+\frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right)+\text{C}$$

$$= \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+\text{C}$$

Question 8:

$\sqrt{x^2+3x}$

Answer

Let $I = \int\sqrt{x^2+3x}\,dx$

$$= \int\sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}\,dx$$

$$= \int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\,dx$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x}-\frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

$$=\frac{(2x+3)}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

Question 9:

$\sqrt{1+\frac{x^2}{9}}$

Answer

Let $I=\int\sqrt{1+\frac{x^2}{9}}dx=\frac{1}{3}\int\sqrt{9+x^2}\,dx=\frac{1}{3}\int\sqrt{(3)^2+x^2}\,dx$

It is known that, $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$\therefore I=\frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right]+C$$

$$=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$$

Question 10:

$\int\sqrt{1+x^2}\,dx$ is equal to

A. $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

B. $\frac{2}{3}\left(1+x^2\right)^{\frac{2}{3}}+C$

C. $\frac{2}{3}x\left(1+x^2\right)^{\frac{3}{2}}+C$

D. $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

Answer

It is known that, $\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$\therefore \int\sqrt{1+x^2}\,dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$$

Hence, the correct Answer is A.

Question 11:

$\int\sqrt{x^2-8x+7}\,dx$ is equal to

A. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Answer

$$\begin{aligned}\text{Let } I &= \int\sqrt{x^2-8x+7}\,dx\\ &= \int\sqrt{\left(x^2-8x+16\right)-9}\,dx\\ &= \int\sqrt{(x-4)^2-(3)^2}\,dx\end{aligned}$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$$

Hence, the correct Answer is D.

Exercise 7.8

Question 1:

$$\int_a^b x\,dx$$

Answer

It is known that,

$$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f\left(a+(n-1)h\right)\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=a,\ b=b,\text{and } f(x)=x$

$$\therefore\int_a^b x\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{a+a+a+\ldots+a}_{n\text{ times}}\right)+\left(h+2h+3h+\ldots+(n-1)h\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(1+2+3+\ldots+(n-1)\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$

$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\left(b^2-a^2\right)$$

Question 2:

$$\int_0^5 (x+1)\,dx$$

Answer

Let $I = \int_0^5 (x+1)\,dx$

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)\ldots f\left(a+(n-1)h\right)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=5$, and $f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)\,dx = (5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots+f\left((n-1)\frac{5}{n}\right)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+1\ldots1}_{n\text{ times}}\right)+\left[\frac{5}{n}+2\cdot\frac{5}{n}+3\cdot\frac{5}{n}+\ldots(n-1)\frac{5}{n}\right]\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3\ldots(n-1)\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\cdot\frac{(n-1)n}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$=5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=5\left[1+\frac{5}{2}\right]$$

$$=5\left[\frac{7}{2}\right]$$

$$=\frac{35}{2}$$

Question 3:

$\int_2^3 x^2 dx$

Answer

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)\dots f\{a+(n-1)h\}\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=2$, $b=3$, and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2 dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)\dots f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\dots\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2\cdot2\cdot\frac{1}{n}\right\}+\dots+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2\cdot2\cdot\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{2^2+\dots+2^2}_{n\text{ times}}\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\dots+\left(\frac{n-1}{n}\right)^2\right\}+2\cdot2\cdot\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\dots+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2\dots+(n-1)^2\right\}+\frac{4}{n}\left\{1+2+\dots+(n-1)\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$=\lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$=4+\frac{2}{6}+2$$

$$=\frac{19}{3}$$

Question 4:

$$\int_1^4 \left(x^2 - x\right) dx$$

Answer

$$\text{Let } I = \int_1^4 \left(x^2 - x\right) dx$$

$$= \int_1^4 x^2\, dx - \int_1^4 x\, dx$$

$$\text{Let } I = I_1 - I_2, \text{ where } I_1 = \int_1^4 x^2 dx \text{ and } I_2 = \int_1^4 x\, dx \qquad \ldots(1)$$

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + f\left(a+(n-1)h\right)\right], \text{ where } h = \frac{b-a}{n}$$

$$\text{For } I_1 = \int_1^4 x^2 dx,$$

$$a = 1, b = 4, \text{ and } f(x) = x^2$$

$$\therefore h = \frac{4-1}{n} = \frac{3}{n}$$

$$I_1 = \int_1^4 x^2 dx = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1) + f(1+h) + \ldots + f\left(1+(n-1)h\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2 + \left(1+\frac{3}{n}\right)^2 + \left(1+2\cdot\frac{3}{n}\right)^2 + \ldots\left(1+\frac{(n-1)3}{n}\right)^2\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2 + \left\{1^2 + \left(\frac{3}{n}\right)^2 + 2\cdot\frac{3}{n}\right\} + \ldots + \left\{1^2 + \left(\frac{(n-1)3}{n}\right)^2 + \frac{2\cdot(n-1)\cdot 3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1^2 + \ldots + 1^2}_{n \text{ times}}\right) + \left(\frac{3}{n}\right)^2\left\{1^2 + 2^2 + \ldots + (n-1)^2\right\} + 2\cdot\frac{3}{n}\left\{1 + 2 + \ldots + (n-1)\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$

$$= 3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$

$$= 3[1+3+3]$$

$$= 3[7]$$

$$I_1 = 21 \qquad \ldots(2)$$

For $I_2 = \int_1^4 x\,dx$,

$a = 1, b = 4,$ and $f(x) = x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\therefore I_2 = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots f(a+(n-1)h)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+(1+h)+\ldots+(1+(n-1)h)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+\ldots+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+\ldots+1}_{n\text{ times}}\right)+\frac{3}{n}\left(1+2+\ldots+(n-1)\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{3}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= 3\left[1+\frac{3}{2}\right]$$

$$= 3\left[\frac{5}{2}\right]$$

$$I_2 = \frac{15}{2} \qquad \ldots(3)$$

From equations (2) and (3), we obtain

$$I = I_1 + I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 5:

$$\int_{-1}^{1} e^x dx$$

Answer

$$\text{Let } I = \int_{-1}^{1} e^x dx \qquad \ldots(1)$$

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h)\ldots f(a+(n-1)h)\right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = -1$, $b = 1$, and $f(x) = e^x$

$$\therefore h = \frac{1+1}{n} = \frac{2}{n}$$

$$\therefore I=(1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+...+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\cdot\frac{2}{n}\right)}+...e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$$

$$=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{2}-1}{e^{\frac{2}{n}}-1}\right]$$

$$=\frac{e^{-1}\times 2\left(e^{2}-1\right)}{\lim\limits_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}$$

$$=e^{-1}\left[\frac{2\left(e^{2}-1\right)}{2}\right] \qquad \left[\lim_{h\to 0}\left(\frac{e^{h}-1}{h}\right)=1\right]$$

$$=\frac{e^{2}-1}{e}$$

$$=\left(e-\frac{1}{e}\right)$$

Question 6:

$$\int_0^4\left(x+e^{2x}\right)dx$$

Answer

It is known that,

$$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f\left(a+(n-1)h\right)\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=4$, and $f(x)=x+e^{2x}$

$$\therefore h=\frac{4-0}{n}=\frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x+e^{2x}\right)dx = (4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+\ldots+f((n-1)h)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^{0}\right)+\left(h+e^{2h}\right)+\left(2h+e^{2\cdot 2h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+\ldots+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+\ldots(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^{8}-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$= 4(2)+4\lim_{n\to\infty}\frac{\left(e^{8}-1\right)}{\left(\dfrac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$= 8+\frac{4\cdot\left(e^{8}-1\right)}{8} \qquad \left(\lim_{x\to 0}\frac{e^{x}-1}{x}=1\right)$$

$$= 8+\frac{e^{8}-1}{2}$$

$$= \frac{15+e^{8}}{2}$$

Exercise 7.9

Question 1:

$\int_{-1}^{1}(x+1)dx$

Answer

Let $I=\int_{-1}^{1}(x+1)dx$

$\int(x+1)\,dx=\frac{x^2}{2}+x=\mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned}I&=\mathrm{F}(1)-\mathrm{F}(-1)\\&=\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)\\&=\frac{1}{2}+1-\frac{1}{2}+1\\&=2\end{aligned}$$

Question 2:

$\int_{2}^{3}\frac{1}{x}dx$

Answer

Let $I=\int_{2}^{3}\frac{1}{x}\,dx$

$\int\frac{1}{x}dx=\log|x|=\mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned}I&=\mathrm{F}(3)-\mathrm{F}(2)\\&=\log|3|-\log|2|=\log\frac{3}{2}\end{aligned}$$

Question 3:

$\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

Answer

Let $I = \int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

$$\int \left(4x^3 - 5x^2 + 6x + 9\right) dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x)$$

$$= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(2) - \mathrm{F}(1)$$

$$I = \left\{2^4 - \frac{5\cdot(2)^3}{3} + 3(2)^2 + 9(2)\right\} - \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + 3 + 9\right)$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

Question 4:

$\int_0^{\frac{\pi}{4}} \sin 2x dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \sin 2x \, dx$

$$\int \sin 2x \, dx = \left(\frac{-\cos 2x}{2}\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\mathrm{F}\left(\frac{\pi}{4}\right)-\mathrm{F}(0)$$

$$=-\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right)-\cos 0\right]$$

$$=-\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right)-\cos 0\right]$$

$$=-\frac{1}{2}[0-1]$$

$$=\frac{1}{2}$$

Question 5:

$$\int_0^{\frac{\pi}{2}}\cos 2x\,dx$$

Answer

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\cos 2x\,dx$$

$$\int\cos 2x\,dx=\left(\frac{\sin 2x}{2}\right)=\mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\mathrm{F}\left(\frac{\pi}{2}\right)-\mathrm{F}(0)$$

$$=\frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right)-\sin 0\right]$$

$$=\frac{1}{2}[\sin\pi-\sin 0]$$

$$=\frac{1}{2}[0-0]=0$$

Question 6:

$$\int_4^5 e^x\,dx$$

Answer

Let $I = \int_4^5 e^x dx$

$\int e^x dx = e^x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$I = \mathrm{F}(5) - \mathrm{F}(4)$

$= e^5 - e^4$

$= e^4(e-1)$

Question 7:

$\int_0^{\frac{\pi}{4}} \tan x \, dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$\int \tan x \, dx = -\log|\cos x| = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$I = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}(0)$

$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$

$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$

$= -\log(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\log 2$

Question 8:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

Answer

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

$\int \operatorname{cosec} x \; dx = \log\left|\operatorname{cosec} x - \cot x\right| = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}\left(\frac{\pi}{6}\right)$$
$$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$
$$= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|$$
$$= \log\left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right)$$

Question 9:

$$\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$$

Answer

Let $I = \int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$
$$= \sin^{-1}(1) - \sin^{-1}(0)$$
$$= \frac{\pi}{2} - 0$$
$$= \frac{\pi}{2}$$

Question 10:

$$\int_0^1 \frac{dx}{1 + x^2}$$

Answer

$$\text{Let } I = \int_0^1 \frac{dx}{1+x^2}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$
$$= \tan^{-1}(1) - \tan^{-1}(0)$$
$$= \frac{\pi}{4}$$

Question 11:

$$\int_2^3 \frac{dx}{x^2-1}$$

Answer

$$\text{Let } I = \int_2^3 \frac{dx}{x^2-1}$$

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(3) - \mathrm{F}(2)$$
$$= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$$
$$= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right]$$
$$= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$$
$$= \frac{1}{2}\left[\log\frac{3}{2}\right]$$

Question 12:

$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

$\int \cos^2 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \left[\mathrm{F}\left(\frac{\pi}{2}\right) - \mathrm{F}(0)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right]$$

$$= \frac{\pi}{4}$$

Question 13:

$\int_2^3 \frac{x\,dx}{x^2+1}$

Answer

Let $I = \int_2^3 \frac{x}{x^2+1} dx$

$\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2} \log\left(1+x^2\right) = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(3) - \mathrm{F}(2)$$

$$= \frac{1}{2}\left[\log\left(1+(3)^2\right) - \log\left(1+(2)^2\right)\right]$$

$$= \frac{1}{2}\left[\log(10) - \log(5)\right]$$

$$= \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2$$

Question 14:

$$\int_0^1 \frac{2x+3}{5x^2+1}dx$$

Answer

$$\text{Let } I = \int_0^1 \frac{2x+3}{5x^2+1}dx$$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}x\right)$$

$$= \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$

$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question 15:

$$\int_0^1 x e^{x^2} dx$$

Answer

Let $I = \int_0^1 xe^{x^2} dx$

Put $x^2 = t \Rightarrow 2x\ dx = dt$

As $x \to 0, t \to 0$ and as $x \to 1, t \to 1$,

$\therefore I = \frac{1}{2}\int_0^1 e^t dt$

$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \mathrm{F}(t)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$
$$= \frac{1}{2}e - \frac{1}{2}e^0$$
$$= \frac{1}{2}(e-1)$$

Question 16:

$\int_0^1 \frac{5x^2}{x^2+4x+3}$

Answer

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3} dx$

Dividing $5x^2$ by x^2+4x+3, we obtain

$$I = \int_1^2 \left\{5 - \frac{20x+15}{x^2+4x+3}\right\} dx$$
$$= \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3} dx$$
$$= [5x]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3} dx$$

$I = 5 - I_1$, where $I = \int_1^2 \frac{20x+15}{x^2+4x+3} dx$ ...(1)

Consider $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+8}dx$

Let $20x+15 = \text{A}\frac{d}{dx}\left(x^2+4x+3\right)+\text{B}$

$= 2\text{A}x+\left(4\text{A}+\text{B}\right)$

Equating the coefficients of x and constant term, we obtain

A = 10 and B = −25

$\Rightarrow I_1 = 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx - 25\int_1^2 \frac{dx}{x^2+4x+3}$

Let $x^2+4x+3=t$

$\Rightarrow (2x+4)dx = dt$

$\Rightarrow I_1 = 10\int\frac{dt}{t} - 25\int\frac{dx}{(x+2)^2-1^2}$

$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right]$

$= \left[10\log\left(x^2+4x+3\right)\right]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$

$= \left[10\log 15 - 10\log 8\right] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$

$= \left[10\log(5\times3) - 10\log(4\times2)\right] - \frac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$

$= \left[10\log 5 + 10\log 3 - 10\log 4 - 10\log 2\right] - \frac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$

$= \left[10+\frac{25}{2}\right]\log 5 + \left[-10-\frac{25}{2}\right]\log 4 + \left[10-\frac{25}{2}\right]\log 3 + \left[-10+\frac{25}{2}\right]\log 2$

$= \frac{45}{2}\log 5 - \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$

$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$

Substituting the value of I_1 in (1), we obtain

$$I = 5 - \left[\frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}\right]$$

$$= 5 - \frac{5}{2}\left[9\log\frac{5}{4} - \log\frac{3}{2}\right]$$

Question 17:

$$\int_0^{\frac{\pi}{4}}\left(2\sec^2 x + x^3 + 2\right)dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}}\left(2\sec^2 x + x^3 + 2\right)dx$$

$$\int\left(2\sec^2 x + x^3 + 2\right)dx = 2\tan x + \frac{x^4}{4} + 2x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}(0)$$

$$= \left\{\left(2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)\right) - \left(2\tan 0 + 0 + 0\right)\right\}$$

$$= 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

Question 18:

$$\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$$

Answer

Let $I = \int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$

$= -\int_0^{\pi} \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx$

$= -\int_0^{\pi} \cos x \, dx$

$\int \cos x \, dx = \sin x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$I = \mathrm{F}(\pi) - \mathrm{F}(0)$

$= \sin \pi - \sin 0$

$= 0$

Question 19:

$\int_0^2 \frac{6x+3}{x^2+4} dx$

Answer

Let $I = \int_0^2 \frac{6x+3}{x^2+4} dx$

$\int \frac{6x+3}{x^2+4} dx = 3 \int \frac{2x+1}{x^2+4} dx$

$= 3 \int \frac{2x}{x^2+4} dx + 3 \int \frac{1}{x^2+4} dx$

$= 3 \log\left(x^2+4\right) + \frac{3}{2} \tan^{-1} \frac{x}{2} = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = F(2) - F(0)$$
$$= \left\{3\log\left(2^2+4\right) + \frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\} - \left\{3\log(0+4) + \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$$
$$= 3\log 8 + \frac{3}{2}\tan^{-1}1 - 3\log 4 - \frac{3}{2}\tan^{-1}0$$
$$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$$
$$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$$
$$= 3\log 2 + \frac{3\pi}{8}$$

Question 20:

$$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$$

Answer

$$\text{Let } I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$$

$$\int\left(xe^x + \sin\frac{\pi x}{4}\right)dx = x\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx + \left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$
$$= xe^x - \int e^x dx - \frac{4\pi}{\pi}\cos\frac{x}{4}$$
$$= xe^x - e^x - \frac{4\pi}{\pi}\cos\frac{x}{4}$$
$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$

$$= \left(1.e^1 - e^1 - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0.e^0 - e^0 - \frac{4}{\pi}\cos 0\right)$$

$$= e - e - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right) + 1 + \frac{4}{\pi}$$

$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

Question 21:

$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ equals

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$

Answer

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Hence, the correct Answer is D.

Question 22:

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ equals

A. $\frac{\pi}{6}$

B. $\frac{\pi}{12}$

C. $\frac{\pi}{24}$

D. $\frac{\pi}{4}$

Answer

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put $3x = t \Rightarrow 3dx = dt$

$$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$

$$= \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \frac{1}{6}\tan^{-1} 0$$

$$= \frac{1}{6}\tan^{-1} 1 - 0$$

$$= \frac{1}{6}\times\frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Hence, the correct Answer is C.

Exercise 7.10

Question 1:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Answer

$$\int_0^1 \frac{x}{x^2+1}dx$$

$$\text{Let } x^2+1=t \;\Rightarrow\; 2x\,dx=dt$$

When x = 0, t = 1 and when x = 1, t = 2

$$\therefore \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\left[\log|t|\right]_1^2$$

$$= \frac{1}{2}\left[\log 2 - \log 1\right]$$

$$= \frac{1}{2}\log 2$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\,d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^4\phi\cos\phi\,d\phi$$

Also, let $\sin\phi = t \Rightarrow \cos\phi\,d\phi = dt$

When $\phi = 0$, $t = 0$ and when $\phi = \frac{\pi}{2}$, $t = 1$

$$\therefore I = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$

$$= \int_0^1 \left[t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right]dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{11}{2}}}{\frac{11}{2}}-\frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3}+\frac{2}{11}-\frac{4}{7}$$

$$= \frac{154+42-132}{231}$$

$$= \frac{64}{231}$$

Question 3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Answer

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta \cdot \sec^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \theta \cdot \sec^2\theta\, d\theta$$

Takingθas first function and $\sec^2\theta$ as second function and integrating by parts, we obtain

$$I = 2\left[\theta \int \sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int \sec^2\theta\, d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int \tan\theta\, d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

Question 4:

$$\int_0^2 x\sqrt{x+2}\ \left(\text{Put } x+2 = t^2\right)$$

Answer

$$\int_0^2 x\sqrt{x+2}\,dx$$

Let x + 2 = t^2 ⇒ dx = 2tdt

When x = 0, $t = \sqrt{2}$ and when x = 2, t = 2

$$\therefore \int_0^2 x\sqrt{x+2}\,dx = \int_{\sqrt{2}}^2 \left(t^2-2\right)\sqrt{t^2}\,2t\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^2-2\right)t^2\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^4-2t^2\right)dt$$

$$= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$= 2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$

$$= 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$

$$= 2\left[\frac{16+8\sqrt{2}}{15}\right]$$

$$= \frac{16\left(2+\sqrt{2}\right)}{15}$$

$$= \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$$

Let cos x = t ⇒ −sinx dx = dt

When x = 0, t = 1 and when $x = \frac{\pi}{2},\ t = 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{dt}{1+t^2}$$

$$= -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[-\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Answer

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let $x-\frac{1}{2}=t$ $\Rightarrow$ dx = dt

When $x=0,\ t=-\frac{1}{2}$ and when $x=2,\ t=\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}=\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$$

$$=\left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)}\log\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}-\frac{\log\frac{\sqrt{17}}{2}-\frac{1}{2}}{\log\frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3}-\log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$=\frac{1}{\sqrt{17}}\log\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$$

Question 7:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}$$

Answer

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{\left(x^2+2x+1\right)+4}=\int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}$$

Let x + 1 = t ⇒ dx = dt

When x = −1, t = 0 and when x = 1, t = 2

$$\therefore \int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}=\int_{0}^{2}\frac{dt}{t^2+2^2}$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$$

Question 8:

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Answer

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Let 2x = t ⇒ 2dx = dt

When x = 1, t = 2 and when x = 2, t = 4

$$\therefore \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \frac{1}{2} \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t dt$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\Rightarrow \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt = \int_2^4 e^t \left[f(t) + f'(t)\right] dt$$

$$= \left[e^t f(t)\right]_2^4$$

$$= \left[e^t \cdot \frac{2}{t}\right]_2^4$$

$$= \left[\frac{e^t}{t}\right]_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^2\left(e^2 - 2\right)}{4}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$ is

A. 6

B. 0

C. 3

D. 4

Answer

Let $I = \int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$

Also, let $x = \sin\theta \Rightarrow dx = \cos\theta \, d\theta$

When $x = \frac{1}{3}$, $\theta = \sin^{-1}\left(\frac{1}{3}\right)$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta - \sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(1 - \sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^2\theta \sin^2\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\cos\theta\right)^{\frac{5}{3}}}{\left(\sin\theta\right)^{\frac{5}{3}}} \operatorname{cosec}^2\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\cot\theta\right)^{\frac{5}{3}} \operatorname{cosec}^2\theta \, d\theta$$

Let $\cot\theta = t \Rightarrow -\operatorname{cosec}2\theta \, d\theta = dt$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

$$\therefore I = -\int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$$

$$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$= \frac{3}{8}[16]$$

$$= 3 \times 2$$

$$= 6$$

Hence, the correct Answer is A.

Question 10:

If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is

A. cos x + x sin x

B. x sin x

C. x cos x

D. sin x + x cos x

Answer

$$f(x) = \int_0^x t \sin t dt$$

Integrating by parts, we obtain

$$f(x) = t\int_0^x \sin t\,dt - \int_0^x \left\{\left(\frac{d}{dt}t\right)\int \sin t\,dt\right\}dt$$
$$= \left[t(-\cos t)\right]_0^x - \int_0^x (-\cos t)\,dt$$
$$= \left[-t\cos t + \sin t\right]_0^x$$
$$= -x\cos x + \sin x$$

$$\Rightarrow f'(x) = -\left[\{x(-\sin x)\} + \cos x\right] + \cos x$$
$$= x\sin x - \cos x + \cos x$$
$$= x\sin x$$

Hence, the correct Answer is B.

Exercise 7.11

Question 1:

$$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$$

Answer

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx \qquad ...(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right)dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ...(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos}}{\sqrt{\cos}+\sqrt{\sin x}}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 3:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}$$

Answer

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx \qquad \ldots(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1\cdot dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

Question 4:

$$\int_0^{\frac{\pi}{2}}\frac{\cos^5 x\,dx}{\sin^5 x+\cos^5 x}$$

Answer

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{\cos^5 x}{\sin^5 x+\cos^5 x}dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1\cdot dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

Question 5:

$$\int_{-5}^{5}|x+2|\,dx$$

Answer

$$\text{Let } I=\int_{-5}^{5}|x+2|\,dx$$

It can be seen that (x + 2) ≤ 0 on [−5, −2] and (x + 2) ≥ 0 on [−2, 5].

$$\therefore I=\int_{-5}^{-2}-(x+2)dx+\int_{-2}^{5}(x+2)\,dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$$

$$I=-\left[\frac{x^2}{2}+2x\right]_{-5}^{-2}+\left[\frac{x^2}{2}+2x\right]_{-2}^{5}$$

$$=-\left[\frac{(-2)^2}{2}+2(-2)-\frac{(-5)^2}{2}-2(-5)\right]+\left[\frac{(5)^2}{2}+2(5)-\frac{(-2)^2}{2}-2(-2)\right]$$

$$=-\left[2-4-\frac{25}{2}+10\right]+\left[\frac{25}{2}+10-2+4\right]$$

$$=-2+4+\frac{25}{2}-10+\frac{25}{2}+10-2+4$$

$$=29$$

Question 6:

$$\int_2^8 |x-5|\,dx$$

Answer

Let $I=\int_2^8 |x-5|\,dx$

It can be seen that $(x-5)\le 0$ on [2, 5] and $(x-5)\ge 0$ on [5, 8].

$$I=\int_2^5-(x-5)\,dx+\int_5^8(x-5)\,dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$$

$$=-\left[\frac{x^2}{2}-5x\right]_2^5+\left[\frac{x^2}{2}-5x\right]_5^8$$

$$=-\left[\frac{25}{2}-25-2+10\right]+\left[32-40-\frac{25}{2}+25\right]$$

$$=9$$

Question 7:

$$\int_0^1 x(1-x)^n\,dx$$

Answer

Let $I = \int_0^1 x(1-x)^n \, dx$

$\therefore I = \int_0^1 (1-x)(1-(1-x))^n \, dx$

$= \int_0^1 (1-x)(x)^n \, dx$

$= \int_0^1 \left(x^n - x^{n+1}\right) dx$

$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$= \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$

$= \frac{(n+2)-(n+1)}{(n+1)(n+2)}$

$= \frac{1}{(n+1)(n+2)}$

Question 8:

$\int_0^{\frac{\pi}{4}} \log\left(1+\tan x\right) dx$

Answer

Let $I=\int_0^{\frac{\pi}{4}}\log\left(1+\tan x\right)dx$ $\quad\ldots(1)$

$\therefore I=\int_0^{\frac{\pi}{4}}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$ $\quad\left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left\{1+\frac{1-\tan x}{1+\tan}\right\}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\frac{2}{\left(1+\tan x\right)}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log 2\,dx-\int_0^{\frac{\pi}{4}}\log\left(1+\tan x\right)dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log 2dx-I$ $\quad\left[\text{From }(1)\right]$

$\Rightarrow 2I=\left[x\log 2\right]_0^{\frac{\pi}{4}}$

$\Rightarrow 2I=\frac{\pi}{4}\log 2$

$\Rightarrow I=\frac{\pi}{8}\log 2$

Question 9:

$\int_0^2 x\sqrt{2-x}dx$

Answer

Let $I=\int_0^2 x\sqrt{2-x}\,dx$

$I=\int_0^2 (2-x)\sqrt{x}\,dx \qquad \left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$

$=\int_0^2\left\{2x^{\frac{1}{2}}-x^{\frac{3}{2}}\right\}dx$

$=\left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$

$=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\right]_0^2$

$=\frac{4}{3}(2)^{\frac{3}{2}}-\frac{2}{5}(2)^{\frac{5}{2}}$

$=\frac{4\times 2\sqrt{2}}{3}-\frac{2}{5}\times 4\sqrt{2}$

$=\frac{8\sqrt{2}}{3}-\frac{8\sqrt{2}}{5}$

$=\frac{40\sqrt{2}-24\sqrt{2}}{15}$

$=\frac{16\sqrt{2}}{15}$

Question 10:

$\int_0^{\frac{\pi}{2}}\left(2\log\sin x-\log\sin 2x\right)dx$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log(2\sin x \cos x)\}\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log \sin x - \log \cos x - \log 2\}\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \sin x - \log \cos x - \log 2\}\, dx \quad ...(1)$$

$$\text{It is known that, } \left(\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \cos x - \log \sin x - \log 2\}\, dx \quad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2)\, dx$$

$$\Rightarrow 2I = -2\log 2 \int_0^{\frac{\pi}{2}} 1\, dx$$

$$\Rightarrow I = -\log 2\left[\frac{\pi}{2}\right]$$

$$\Rightarrow I = \frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I = \frac{\pi}{2}\left[\log \frac{1}{2}\right]$$

$$\Rightarrow I = \frac{\pi}{2}\log \frac{1}{2}$$

Question 11:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$$

Answer

$$\text{Let } I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$$

As $\sin^2(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x$, therefore, $\sin^2 x$ is an even function.

It is known that if f(x) is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$$I = 2\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

Question 12:

$$\int_0^{\pi} \frac{x\,dx}{1+\sin x}$$

Answer

$$\text{Let } I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)}\,dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x}\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\pi}\frac{\pi}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\left\{\sec^2 x-\tan x\sec x\right\}dx$$

$$\Rightarrow 2I=\pi\left[\tan x-\sec x\right]_0^{\pi}$$

$$\Rightarrow 2I=\pi[2]$$

$$\Rightarrow I=\pi$$

Question 13:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7 x\,dx$$

Answer

$$\text{Let } I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7 x dx \qquad \ldots(1)$$

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, therefore, $\sin^2 x$ is an odd function.

It is known that, if f(x) is an odd function, then $\int_{-a}^{a} f(x)dx=0$

$$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7 x\,dx=0$$

Question 14:

$$\int_0^{2\pi}\cos^5 x dx$$

Answer

$$\text{Let } I=\int_0^{2\pi}\cos^5 x dx \qquad \ldots(1)$$

$$\cos^5(2\pi-x)=\cos^5 x$$

It is known that,

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)$$
$$= 0 \text{ if } f(2a-x) = -f(x)$$

$$\therefore I = 2\int_0^{\pi} \cos^5 x\,dx$$

$$\Rightarrow I = 2(0) = 0 \qquad \left[\cos^5(\pi - x) = -\cos^5 x\right]$$

Question 15:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left(\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x\cos x}\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{1+\sin x\cos x}\,dx$$
$$\Rightarrow I = 0$$

Question 16:

$$\int_0^{\pi} \log(1+\cos x)\,dx$$

Answer

$$\text{Let } I = \int_0^{\pi} \log(1+\cos x)\,dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\pi} \log\left(1+\cos(\pi - x)\right)dx \qquad \left(\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \log(1-\cos x)\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \left\{ \log(1+\cos x) + \log(1-\cos x) \right\} dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log\left(1-\cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x\, dx$$

$$\Rightarrow 2I = 2\int_0^{\pi} \log \sin x\, dx$$

$$\Rightarrow I = \int_0^{\pi} \log \sin x\, dx \qquad \ldots(3)$$

$\sin(\pi - x) = \sin x$

$$\therefore I = 2\int_0^{\frac{\pi}{2}} \log \sin x\, dx \qquad \ldots(4)$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = 2\int_0^{\frac{\pi}{2}} \log \cos x\, dx \qquad \ldots(5)$$

Adding (4) and (5), we obtain

$$2I = 2\int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x + \log 2 - \log 2\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log 2 \sin x \cos x - \log 2\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \int_0^{\frac{\pi}{2}} \log 2\, dx$$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}, \pi =$

$$\therefore I = \frac{1}{2}\int_0^{\pi} \log \sin t\, dt - \frac{\pi}{2}\log 2$$

$$\Rightarrow I = \frac{1}{2} I - \frac{\pi}{2}\log 2$$

$$\Rightarrow \frac{I}{2} = -\frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\pi \log 2$$

Question 17:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$

Answer

$$\text{Let } I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \qquad ...(1)$$

It is known that, $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

$$\Rightarrow 2I = \int_0^a 1\cdot dx$$

$$\Rightarrow 2I = [x]_0^a$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

Question 18:

$$\int_0^4 |x-1|\,dx$$

Answer

$$I = \int_0^4 |x-1|\,dx$$

It can be seen that, $(x - 1) \le 0$ when $0 \le x \le 1$ and $(x - 1) \ge 0$ when $1 \le x \le 4$

$$I=\int_0^1 |x-1|dx+\int_1^4 |x-1|dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$$

$$=\int_0^1 -(x-1)dx+\int_0^4 (x-1)dx$$

$$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^4$$

$$=1-\frac{1}{2}+\frac{(4)^2}{2}-4-\frac{1}{2}+1$$

$$=1-\frac{1}{2}+8-4-\frac{1}{2}+1$$

$$=5$$

Question 19:

Show that $\int_0^a f(x)g(x)dx=2\int_0^a f(x)dx,$ if f and g are defined as $f(x)=f(a-x)$ and $g(x)+g(a-x)=4$

Answer

$$\text{Let } I=\int_0^a f(x)g(x)dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^a f(a-x)g(a-x)dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^a f(x)g(a-x)dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^a \{f(x)g(x)+f(x)g(a-x)\}dx$$

$$\Rightarrow 2I=\int_0^a f(x)\{g(x)+g(a-x)\}dx$$

$$\Rightarrow 2I=\int_0^a f(x)\times 4\,dx \qquad [g(x)+g(a-x)=4]$$

$$\Rightarrow I=2\int_0^a f(x)dx$$

Question 20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5 x+1\right)dx$ is

A. 0

B. 2

C. π

D. 1

Answer

$$\text{Let } I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5 x+1\right)dx$$

$$\Rightarrow I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x^3dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tan^5 x dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1\cdot dx$$

It is known that if f(x) is an even function, then $\int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx$ and

if f(x) is an odd function, then $\int_{-a}^{a}f(x)dx=0$

$$I=0+0+0+2\int_0^{\frac{\pi}{2}}1\cdot dx$$

$$=2\left[x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{2\pi}{2}$$

$$\pi=$$

Hence, the correct Answer is C.

Question 21:

The value of $\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

A. 2

B. $\frac{3}{4}$

C. 0

D. -2

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \qquad ...(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left[\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right]dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left\{\log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \log\left(\frac{4+3\cos x}{4+3\sin x}\right)\right\}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1\,dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0\,dx$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is C.

Miscellaneous Solutions

Question 1:

$$\frac{1}{x-x^3}$$

Answer

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}=\frac{1}{x(1-x)(1+x)}$$

$$\text{Let } \frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x} \qquad ...(1)$$

$$\Rightarrow 1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain $-A + B - C = 0$

$B + C = 0$

$A = 1$

On solving these equations, we obtain

$$A=1, B=\frac{1}{2}, \text{ and } C=-\frac{1}{2}$$

From equation (1), we obtain

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

$$\Rightarrow \int \frac{1}{x(1-x)(1+x)} dx = \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{1-x} dx - \frac{1}{2} \int \frac{1}{1+x} dx$$

$$= \log|x| - \frac{1}{2} \log|(1-x)| - \frac{1}{2} \log|(1+x)|$$

$$= \log|x| - \log\left|(1-x)^{\frac{1}{2}}\right| - \log\left|(1+x)^{\frac{1}{2}}\right|$$

$$= \log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right| + C$$

$$= \log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right| + C$$

$$= \frac{1}{2} \log\left|\frac{x^2}{1-x^2}\right| + C$$

Question 2:

$$\frac{1}{\sqrt{x+a} + \sqrt{(x+b)}}$$

Answer

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$
$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$
$$=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow \int\frac{1}{\sqrt{x+a}-\sqrt{x+b}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$
$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$
$$=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+\text{C}$$

Question 3:

$\frac{1}{x\sqrt{ax-x^2}}$ [Hint: Put $x=\frac{a}{t}$]

Answer

$$\frac{1}{x\sqrt{ax-x^2}}$$

$$\text{Let } x=\frac{a}{t} \Rightarrow \; dx=-\frac{a}{t^2}dt$$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx=\int \frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$=-\int \frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$=-\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$=-\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+\mathrm{C}$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+\mathrm{C}$$

$$=-\frac{2}{a}\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)+\mathrm{C}$$

$$=-\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+\mathrm{C}$$

Question 4:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Answer

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we obtain

$$\frac{x^{-3}}{x^2\cdot x^{-3}\left(x^4+1\right)^{\frac{3}{4}}}=\frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2\cdot x^{-3}}$$

$$=\frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5\cdot\left(x^4\right)^{-\frac{3}{4}}}$$

$$=\frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$=\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4}=t \Rightarrow -\frac{4}{x^5}dx=dt\Rightarrow\frac{1}{x^5}dx=-\frac{dt}{4}$

$$\therefore\int\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx=\int\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$=-\frac{1}{4}\int(1+t)^{-\frac{3}{4}}dt$$

$$=-\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$=-\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Question 5:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\left[\text{Hint: }\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}\text{Put } x=t^6\right]$$

Answer

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

$$\text{Let } x=t^6 \Rightarrow dx=6t^5\,dt$$

$$\therefore \int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=\int\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx$$

$$=\int\frac{6t^5}{t^2(1+t)}dt$$

$$=6\int\frac{t^3}{(1+t)}dt$$

On dividing, we obtain

$$\int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=6\int\left\{\left(t^2-t+1\right)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

Question 6:

$$\frac{5x}{(x+1)\left(x^2+9\right)}$$

Answer

$$\text{Let } \frac{5x}{(x+1)(x^2+9)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+9)} \qquad ...(1)$$

$$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$$

$$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

B + C = 5

9A + C = 0

On solving these equations, we obtain

$$A = -\frac{1}{2}, B = \frac{1}{2}, \text{ and } C = \frac{9}{2}$$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int \frac{5x}{(x+1)(x^2+9)}dx = \int \left\{\frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)}\right\}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{2}\int \frac{x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + \text{C}$$

Question 7:

$$\frac{\sin x}{\sin(x-a)}$$

Answer

$$\frac{\sin x}{\sin(x-a)}$$

Let x – a = t ⇒ dx = dt

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t}dt$$

$$= \int (\cos a + \cot t \sin a)dt$$

$$= t\cos a + \sin a \log|\sin t| + C_1$$

$$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C_1$$

$$= x\cos a + \sin a \log|\sin(x-a)| - a\cos a + C_1$$

$$= \sin a \log|\sin(x-a)| + x\cos a + C$$

Question 8:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Answer

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)}$$

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int x^2 dx = \frac{x^3}{3} + C$$

Question 9:

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Answer

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Let sin x = t ⇒ cos x dx = dt

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \frac{dt}{\sqrt{(2)^2-(t)^2}}$$

$$= \sin^{-1}\left(\frac{t}{2}\right)+C$$

$$= \sin^{-1}\left(\frac{\sin x}{2}\right)+C$$

Question 10:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$$

Answer

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x} = \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$$

$$= \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{\left(\sin^2 x-\sin^2 x\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\cos^2 x\right)}$$

$$= \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{\sin^2 x\left(1-\cos^2 x\right)+\cos^2 x\left(1-\sin^2 x\right)}$$

$$= \frac{-\left(\sin^4 x+\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$

$$= -\cos 2x$$

$$\therefore \int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x} dx = \int -\cos 2x\, dx = -\frac{\sin 2x}{2}+C$$

Question 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Answer

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we obtain

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cdot\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$

$$=\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x+a)\right|+\log\left|\cos(x+b)\right|\right]+\mathrm{C}$$

$$=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+\mathrm{C}$$

Question 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Answer

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t \Rightarrow 4x^3\,dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{1}{4}\sin^{-1} t + C$$
$$= \frac{1}{4}\sin^{-1}\left(x^4\right) + C$$

Question 13:

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Answer

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Let $e^x = t \Rightarrow e^x\,dx = dt$

$$\Rightarrow \int \frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}dx = \int \frac{dt}{(t+1)(t+2)}$$
$$= \int \left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right]dt$$
$$= \log|t+1| - \log|t+2| + C$$
$$= \log\left|\frac{t+1}{t+2}\right| + C$$
$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

Question 14:

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}$$

Answer

$$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

A + C = 0

B + D = 0

4A + C = 0

4B + D = 1

On solving these equations, we obtain

$$A = 0,\ B = \frac{1}{3},\ C = 0, \text{and } D = -\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$$

$$\int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + \text{C}$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + \text{C}$$

Question 15:

$\cos^3 x e^{\log \sin x}$

Answer

$\cos^3 x e^{\log \sin x} = \cos^3 x \times \sin x$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\Rightarrow \int \cos^3 x\, e^{\log \sin x} dx = \int \cos^3 x \sin x dx$$
$$= -\int t \cdot dt$$
$$= -\frac{t^4}{4} + \text{C}$$
$$= -\frac{\cos^4 x}{4} + \text{C}$$

Question 16:

$e^{3\log x}\left(x^4+1\right)^{-1}$

Answer

$$e^{3\log x}\left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

Let $x^4 + 1 = t \ \Rightarrow \ 4x^3\, dx = dt$

$$\Rightarrow \int e^{3\log x}\left(x^4+1\right)^{-1} dx = \int \frac{x^3}{\left(x^4+1\right)} dx$$
$$= \frac{1}{4}\int \frac{dt}{t}$$
$$= \frac{1}{4}\log|t| + \text{C}$$
$$= \frac{1}{4}\log\left|x^4+1\right| + \text{C}$$
$$= \frac{1}{4}\log\left(x^4+1\right) + \text{C}$$

Question 17:

$f'(ax+b)\left[f(ax+b)\right]^n$

Answer

$f'(ax+b)\left[f(ax+b)\right]^n$

Let $f(ax+b)=t \Rightarrow af'(ax+b)dx = dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$

$$= \frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1} + \mathrm{C}$$

Question 18:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

Answer

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x(\sin x\cos\alpha + \cos x\sin\alpha)}}$$

$$= \frac{1}{\sqrt{\sin^4 x\cos\alpha + \sin^3 x\cos x\sin\alpha}}$$

$$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x\sin\alpha}}$$

$$= \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x\sin\alpha}}$$

Let $\cos\alpha + \cot x\sin\alpha = t \Rightarrow -\operatorname{cosec}^2 x\sin\alpha\, dx = dt$

$$\therefore \int \frac{1}{\sin^3 x\sin(x+\alpha)}dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x\sin\alpha}}dx$$

$$= \frac{-1}{\sin\alpha}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{t}\right] + C$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha + \cot x\sin\alpha}\right] + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x\sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x\cos\alpha + \cos x\sin\alpha}{\sin x}} + C$$

$$= -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

Question 19:

$$\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$$

Answer

Let $I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx$

It is known that, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$

$$\Rightarrow I=\int\frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$=\frac{2}{\pi}\int\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

$$=\frac{2}{\pi}\cdot\frac{\pi}{2}\int 1\cdot dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}\,dx$$

$$=x-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}\,dx \qquad ...(1)$$

Let $I_1=\int\cos^{-1}\sqrt{x}\,dx$

Also, let $\sqrt{x}=t \;\Rightarrow\; dx=2t\,dt$

$$\Rightarrow I_1=2\int\cos^{-1}t\cdot t\,dt$$

$$=2\left[\cos^{-1}t\cdot\frac{t^2}{2}-\int\frac{-1}{\sqrt{1-t^2}}\cdot\frac{t^2}{2}dt\right]$$

$$=t^2\cos^{-1}t+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\frac{1-t^2-1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\sqrt{1-t^2}\,dt+\int\frac{1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}-\frac{1}{2}\sin^{-1}t+\sin^{-1}t$$

$$=t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t$$

From equation (1), we obtain

$$I = x - \frac{4}{\pi}\left[t^2\cos t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right]$$

$$= x - \frac{4}{\pi}\left[x\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - \frac{4\pi}{\pi}\left[x\left(\frac{}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\sqrt{x-x^2}}{2} + \frac{}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

Question 20:

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Answer

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$$

Let $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta\, d\theta$

$$I = \int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta$$

$$= -\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta\, d\theta$$

$$= -\int\tan\frac{\theta}{2}\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= -2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta\, d\theta$$

$$= -4\int \sin^2\frac{\theta}{2}\cos\theta\,d\theta$$

$$= -4\int \sin^2\frac{\theta}{2}\cdot\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$= -4\int\left(2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$$

$$= -8\int \sin^2\frac{\theta}{2}\cdot\cos^2\frac{\theta}{2}d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int \sin^2\theta\,d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta+4\int\frac{1-\cos\theta}{2}d\theta$$

$$= -2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]+4\left[\frac{\theta}{2}-\frac{\sin\theta}{2}\right]+C$$

$$= -\theta+\frac{\sin 2\theta}{2}+2\theta-2\sin\theta+C$$

$$= \theta+\frac{\sin 2\theta}{2}-2\sin\theta+C$$

$$= \theta+\frac{2\sin\theta\cos\theta}{2}-2\sin\theta+C$$

$$= \theta+\sqrt{1-\cos^2\theta}\cdot\cos\theta-2\sqrt{1-\cos^2\theta}+C$$

$$= \cos^{-1}\sqrt{x}+\sqrt{1-x}\cdot\sqrt{x}-2\sqrt{1-x}+C$$

$$= -2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+C$$

$$= -2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$$

Question 21:

$$\frac{2+\sin 2x}{1+\cos 2x}e^x$$

Answer

$$I = \int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$$
$$= \int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$$
$$= \int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$$
$$= \int\left(\sec^2 x+\tan x\right)e^x$$

$$\text{Let } f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$$
$$\therefore I = \int\left(f(x)+f'(x)\right]e^x dx$$
$$= e^x f(x)+\text{C}$$
$$= e^x \tan x+\text{C}$$

Question 22:

$$\frac{x^2+x+1}{(x+1)^2(x+2)}$$

Answer

$$\text{Let } \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+2)} \qquad ...(1)$$
$$\Rightarrow x^2+x+1 = A(x+1)(x+2)+B(x+2)+C(x^2+2x+1)$$
$$\Rightarrow x^2+x+1 = A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)$$
$$\Rightarrow x^2+x+1 = (A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

Equating the coefficients of x^2, x,and constant term, we obtain

A + C = 1

3A + B + 2C = 1

2A + 2B + C = 1

On solving these equations, we obtain

A = −2, B = 1, and C = 3

From equation (1), we obtain

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

Question 23:

$$\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Answer

$$I=\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$$

Let $x=\cos\theta \Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta d\theta)$$

$$=-\int\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$$

$$=-\int\tan^{-1}\tan\frac{\theta}{2}\cdot\sin\theta d\theta$$

$$=-\frac{1}{2}\int\theta\cdot\sin\theta d\theta$$

$$=-\frac{1}{2}\left[\theta\cdot(-\cos\theta)-\int1\cdot(-\cos\theta)d\theta\right]$$

$$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]$$

$$=+\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x\cdot x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{x}{2}\cos^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

Question 24:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$$

Answer

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

Let $1+\frac{1}{x^2}=t \Rightarrow \frac{-2}{x^3}dx=dt$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t\,dt$$

$$=-\frac{1}{2}\int t^{\frac{1}{2}}\cdot\log t\,dt$$

Integrating by parts, we obtain

$$I=-\frac{1}{2}\left[\log t\cdot\int t^{\frac{1}{2}}dt-\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+\mathrm{C}$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi}e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Answer

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{\operatorname{cosec}^2\frac{x}{2}}{2} - \cot\frac{x}{2}\right) dx$$

$$\text{Let } f(x) = -\cot\frac{x}{2}$$

$$\Rightarrow f'(x) = -\left(-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(f(x) + f'(x)\right] dx$$

$$= \left[e^x \cdot f(x)\,dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x \cdot \cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^{\pi} \times \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot\frac{\pi}{4}\right]$$

$$= -\left[e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1\right]$$

$$= e^{\frac{\pi}{2}}$$

Question 26:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{(\sin x \cos x)}{\cos^4 x}}{\frac{(\cos^4 x + \sin^4 x)}{\cos^4 x}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

Let $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x \, dx = dt$

When x = 0, t = 0 and when $x = \frac{\pi}{4}, t = 1$

$$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2} \left[\tan^{-1} t \right]_0^1$$

$$= \frac{1}{2} \left[\tan^{-1} 1 - \tan^{-1} 0 \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} \right]$$

$$= \frac{\pi}{8}$$

Question 27:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x \, dx}{\cos^2 x + 4 \sin^2 x}$$

Answer

Let $I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\left(1-\cos^2 x\right)}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x-4}{4-3\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4}{4-3\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}1\,dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4\sec^2 x-3}dx$

$\Rightarrow I=\frac{-1}{3}\left[x\right]_0^{\frac{\pi}{2}}+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4\left(1+\tan^2 x\right)-3}dx$

$\Rightarrow I=-\frac{\pi}{6}+\frac{2}{3}\int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx \qquad ...(1)$

Consider, $\int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx$

Let $2\tan x=t \Rightarrow 2\sec^2 x\,dx=dt$

When $x=0, t=0$ and when $x=\frac{\pi}{2}, t=\infty$

$$\Rightarrow \int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx=\int_0^{\infty}\frac{dt}{1+t^2}$$

$$=\left[\tan^{-1} t\right]_0^{\infty}$$

$$=\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]$$

$$=\frac{\pi}{2}$$

Therefore, from (1), we obtain

$$I=-\frac{\pi}{6}+\frac{2}{3}\left[\frac{\pi}{2}\right]=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\text{Let } I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{-(-\sin 2x)}} dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{-(-1+1-2\sin x\cos x)}} dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{1-(\sin^2 x+\cos^2 x-2\sin x\cos x)}} dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)dx}{\sqrt{1-(\sin x-\cos x)^2}}$$

$$\text{Let } (\sin x-\cos x)=t \;\Rightarrow\; (\sin x+\cos x)dx=dt$$

When $x=\frac{\pi}{6}, t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when $x=\frac{\pi}{3}, t=\left(\frac{\sqrt{3}-1}{2}\right)$

$$I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I=\int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}}=\frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function.

It is known that if f(x) is an even function, then $\int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx$

$$\Rightarrow I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$= \left[2\sin^{-1} t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

Question 29:

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Answer

$$\text{Let } I = \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

$$I = \int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)} \times \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)} dx$$

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$$

$$= \int_0^1 \sqrt{1+x}\, dx + \int_0^1 \sqrt{x}\, dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[(2)^{\frac{3}{2}} - 1\right] + \frac{2}{3}[1]$$

$$= \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2 \cdot 2\sqrt{2}}{3}$$

$$= \frac{4\sqrt{2}}{3}$$

Question 30:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

Also, let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$

When $x = 0,\ t = -1$ and when $x = \frac{\pi}{4},\ t = 0$

$$\Rightarrow (\sin x - \cos x)^2 = t^2$$
$$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$$
$$\Rightarrow 1 - \sin 2x = t^2$$
$$\Rightarrow \sin 2x = 1 - t^2$$

$$\therefore I = \int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)}$$
$$= \int_{-1}^{0} \frac{dt}{9 + 16 - 16t^2}$$
$$= \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \int_{-1}^{0} \frac{dt}{(5)^2 - (4t)^2}$$
$$= \frac{1}{4}\left[\frac{1}{2(5)} \log\left|\frac{5 + 4t}{5 - 4t}\right|\right]_{-1}^{0}$$
$$= \frac{1}{40}\left[\log(1) - \log\left|\frac{1}{9}\right|\right]$$
$$= \frac{1}{40} \log 9$$

Question 31:

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Answer

Let $I=\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)dx=\int_0^{\frac{\pi}{2}}2\sin x\cos x\tan^{-1}(\sin x)dx$

Also, let $\sin x=t \Rightarrow \cos x\,dx=dt$

When $x=0, t=0$ and when $x=\frac{\pi}{2}, t=1$

$$\Rightarrow I=2\int_0^1 t\tan^{-1}(t)dt \qquad \ldots(1)$$

Consider $\int t\cdot\tan^{-1}t\,dt=\tan^{-1}t\cdot\int t\,dt-\int\left\{\frac{d}{dt}\left(\tan^{-1}t\right)\int t\,dt\right\}dt$

$$=\tan^{-1}t\cdot\frac{t^2}{2}-\int\frac{1}{1+t^2}\cdot\frac{t^2}{2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\int\frac{t^2+1-1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\int 1\,dt+\frac{1}{2}\int\frac{1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\cdot t+\frac{1}{2}\tan^{-1}t$$

$$\Rightarrow\int_0^1 t\cdot\tan^{-1}t\,dt=\left[\frac{t^2\cdot\tan^{-1}t}{2}-\frac{t}{2}+\frac{1}{2}\tan^{-1}t\right]_0^1$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-1+\frac{\pi}{4}\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-1\right]=\frac{\pi}{4}-\frac{1}{2}$$

From equation (1), we obtain

$$I=2\left[\frac{\pi}{4}-\frac{1}{2}\right]=\frac{\pi}{2}-1$$

Question 32:

$$\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$$

Answer

$$Let\ I=\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx \quad ...(1)$$

$$I=\int_0^{\pi}\left\{\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}\right\}dx \quad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\pi}\left\{\frac{-(\pi-x)\tan x}{-(\sec x+\tan x)}\right\}dx$$

$$\Rightarrow I=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\tan x}dx \quad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\pi}\frac{\pi\tan x}{\sec x+\tan x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x+1-1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}1.dx-\pi\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi[x]_0^{\pi}-\pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$\Rightarrow 2I=\pi^2-\pi\int_0^{\pi}\left(\sec^2 x-\tan x\sec x\right)dx$$

$$\Rightarrow 2I=\pi^2-\pi[\tan x-\sec x]_0^{\pi}$$

$$\Rightarrow 2I=\pi^2-\pi[\tan\pi-\sec\pi-\tan 0+\sec 0]$$

$$\Rightarrow 2I=\pi^2-\pi[0-(-1)-0+1]$$

$$\Rightarrow 2I=\pi^2-2\pi$$

$$\Rightarrow 2I=\pi(\pi-2)$$

$$\Rightarrow I=\frac{\pi}{2}(\pi-2)$$

Question 33:

$$\int_1^4 \left[|x-1|+|x-2|+|x-3|\right]dx$$

Answer

Let $I = \int_1^4 \left[|x-1|+|x-2|+|x-3|\right]dx$

$$\Rightarrow I = \int_1^4 |x-1|dx + \int_1^4 |x-2|dx + \int_1^4 |x-3|dx$$

$$I = I_1 + I_2 + I_3 \qquad ...(1)$$

where, $I_1 = \int_1^4 |x-1|dx$, $I_2 = \int_1^4 |x-2|dx$, and $I_3 = \int_1^4 |x-3|dx$

$$I_1 = \int_1^4 |x-1|dx$$

$(x-1) \geq 0$ for $1 \leq x \leq 4$

$$\therefore I_1 = \int_1^4 (x-1)dx$$

$$\Rightarrow I_1 = \left[\frac{x^2}{x} - x\right]_1^4$$

$$\Rightarrow I_1 = \left[8-4-\frac{1}{2}+1\right] = \frac{9}{2} \qquad ...(2)$$

$$I_2 = \int_1^4 |x-2|dx$$

$x-2 \geq 0$ for $2 \leq x \leq 4$ and $x-2 \leq 0$ for $1 \leq x \leq 2$

$$\therefore I_2 = \int_1^2 (2-x)dx + \int_2^4 (x-2)dx$$

$$\Rightarrow I_2 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$\Rightarrow I_2 = \left[4-2-2+\frac{1}{2}\right] + \left[8-8-2+4\right]$$

$$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2} \qquad ...(3)$$

$$I_3 = \int_1^4 |x-3|\,dx$$

$x-3 \geq 0$ for $3 \leq x \leq 4$ and $x-3 \leq 0$ for $1 \leq x \leq 3$

$$\therefore I_3 = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx$$

$$\Rightarrow I_3 = \left[3x - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$$

$$\Rightarrow I_3 = [6-4] + \left[\frac{1}{2}\right] = \frac{5}{2} \qquad \ldots(4)$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

Question 34:

$$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$$

Answer

$$\text{Let } I = \int_1^3 \frac{dx}{x^2(x+1)}$$

$$\text{Also, let } \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + C = 0

A + B = 0

B = 1

On solving these equations, we obtain

A = −1, C = 1, and B = 1

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x} \right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence, the given result is proved.

Question 35:

$$\int_0^1 xe^x dx = 1$$

Answer

$$\text{Let } I = \int_0^1 xe^x \; dx$$

Integrating by parts, we obtain

$$I = x\int_0^1 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence, the given result is proved.

Question 36:

$$\int_{-1}^{1} x^{17}\cos^4 x dx = 0$$

Answer

Let $I = \int_{-1}^{1} x^{17}\cos^4 x dx$

Also, let $f(x) = x^{17}\cos^4 x$

$\Rightarrow f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

Therefore, f (x) is an odd function.

It is known that if f(x) is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$\therefore I = \int_{-1}^{1} x^{17}\cos^4 x\,dx = 0$

Hence, the given result is proved.

Question 37:

$$\int_0^{\frac{\pi}{2}} \sin^3 x\,dx = \frac{2}{3}$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x\,dx$

$I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin x\,dx$

$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x\,dx$

$= \int_0^{\frac{\pi}{2}} \sin x\,dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin x\,dx$

$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$

$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$

Hence, the given result is proved.

Question 38:

$$\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx$$

$$I = 2\int_0^{\frac{\pi}{4}} \tan^2 x \tan x\, dx = 2\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)\tan x\, dx$$

$$= 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, dx - 2\int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log\cos x\right]_0^{\frac{\pi}{4}}$$

$$= 1 + 2\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$$

$$= 1 + 2\left[\log\frac{1}{\sqrt{2}} - \log 1\right]$$

$$= 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence, the given result is proved.

Question 39:

$$\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1$$

Answer

$$\text{Let } I = \int_0^1 \sin^{-1} x\, dx$$

$$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 \cdot dx$$

Integrating by parts, we obtain

$$I=\left[\sin^{-1}x\cdot x\right]_0^1-\int_0^1\frac{1}{\sqrt{1-x^2}}\cdot x\,dx$$

$$=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\int_0^1\frac{(-2x)}{\sqrt{1-x^2}}dx$$

Let $1-x^2=t \Rightarrow -2x\,dx=dt$

When $x=0,\ t=1$ and when $x=1,\ t=0$

$$I=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\int_1^0\frac{dt}{\sqrt{t}}$$

$$=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$=\sin^{-1}(1)+\left[-\sqrt{1}\right]$$

$$=\frac{\pi}{2}-1$$

Hence, the given result is proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Answer

Let $I=\int_0^1 e^{2-3x}dx$

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f\left(a+(n-1)h\right)\right]$$

Where, $h=\dfrac{b-a}{n}$

Here, $a=0, b=1,$ and $f(x)=e^{2-3x}$

$$\Rightarrow h=\frac{1-0}{n}=\frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx=(1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+\ldots+f\left(0+(n-1)h\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2+e^{2-3h}+\ldots e^{2-3(n-1)h}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+\dots e^{-3(n-1)h}\right\}\right]$$

$$= \lim_{h\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}(1) \qquad \left[\lim_{n\to\infty}\frac{x}{e^x-1}\right]$$

$$= \frac{-e^{-1}+e^2}{3}$$

$$= \frac{1}{3}\left(e^2-\frac{1}{e}\right)$$

Question 41:

$\int\frac{dx}{e^x+e^{-x}}$ is equal to

A. $\tan^{-1}\left(e^x\right)+C$

B. $\tan^{-1}\left(e^{-x}\right)+C$

C. $\log\left(e^{x}-e^{-x}\right)+C$

D. $\log\left(e^{x}+e^{-x}\right)+C$

Answer

$$\text{Let } I=\int\frac{dx}{e^{x}+e^{-x}}dx=\int\frac{e^{x}}{e^{2x}+1}dx$$

Also, let $e^{x}=t \Rightarrow e^{x}\,dx=dt$

$$\therefore I=\int\frac{dt}{1+t^{2}}$$

$$=\tan^{-1}t+C$$

$$=\tan^{-1}\left(e^{x}\right)+C$$

Hence, the correct Answer is A.

Question 42:

$\int\frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$ is equal to

A. $\frac{-1}{\sin x+\cos x}+C$

B. $\log\left|\sin x+\cos x\right|+C$

C. $\log\left|\sin x-\cos x\right|+C$

D. $\frac{1}{(\sin x+\cos x)^{2}}$

Answer

Let $I = \frac{\cos 2x}{(\cos x + \sin x)^2}$

$$I = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{\cos x - \sin x}{\cos + \sin x} dx$$

Let $\cos x + \sin x = t \Rightarrow (\cos x - \sin x) dx = dt$

$$\therefore I = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|\cos x + \sin x| + C$$

Hence, the correct Answer is B.

Question 43:

If $f(a+b-x) = f(x)$, then $\int_a^b x f(x) dx$ is equal to

A. $\frac{a+b}{2} \int_a^b f(b-x) dx$

B. $\frac{a+b}{2} \int_a^b f(b+x) dx$

C. $\frac{b-a}{2} \int_a^b f(x) dx$

D. $\frac{a+b}{2} \int_a^b f(x) dx$

Answer

Let $I = \int_a^b x f(x) dx \qquad ...(1)$

$$I=\int_a^b (a+b-x)f(a+b-x)\,dx \qquad \left(\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right)$$

$$\Rightarrow I=\int_a^b (a+b-x)f(x)\,dx$$

$$\Rightarrow I=(a+b)\int_a^b f(x)\,dx \quad -I \qquad \left[\text{Using}(1)\right]$$

$$\Rightarrow I+I=(a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow 2I=(a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow I=\left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx$$

Hence, the correct Answer is D.

Question 44:

The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A. 1

B. 0

C. − 1

D. $\frac{\pi}{4}$

Answer

$$\text{Let } I=\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$$

$$\Rightarrow I=\int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$$

$$\Rightarrow I=\int_0^1 \left[\tan^{-1}x-\tan^{-1}(1-x)\right]dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^1 \left[\tan^{-1}(1-x)-\tan^{-1}(1-1+x)\right]dx$$

$$\Rightarrow I=\int_0^1 \left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx$$

$$\Rightarrow I=\int_0^1 \left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^1 \left(\tan^{-1} x + \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x\right) dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is B.